Optimal Dissolution and Structure of Partnerships*

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Abstract

For a partnership model with general type distributions and interdependent values, we derive the optimal dissolution mechanisms that, for arbitrary initial ownership, maximize any convex combination of revenue and social surplus. The solution involves ironing around typically interior worst-off types, which are endogenously determined. We also determine the optimal initial ownership structures. With identical distributions, equal initial shares are always optimal. With non-identical distributions, the optimal initial shares are typically asymmetric, the identity of the agents with large shares may change with the importance of revenue generation, and even fully concentrated initial ownership can be optimal.

Keywords: partnership dissolution, mechanism design, property rights, interdependent values, asymmetric type distributions.

JEL-Classification: D23, D61, D82

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1 Introduction

The Coase Theorem provides the fundamental insight that the connection between the efficiency of the final allocation and the initial ownership structure depends on the ease with which property rights can be reallocated. The final allocation will be efficient irrespective of initial ownership if transaction costs are negligible and property rights are well-defined. By now, there is, however, ample evidence that initial misallocations are not always easily and quickly mended through subsequent transactions, indicating in the light of the Coase Theorem that transaction costs can be substantive.[1] For example, business deadlocks are notoriously costly, with costs ranging from legal expenses to the indirect costs of distraction of partners, and possibly costs due to corruption. Another case in point is land reallocation, which has proved costly and time-consuming even in countries like the United States with a well-functioning legal system and well-defined property rights. It poses major challenges for countries with less well-defined property rights such as China.[2] Likewise, the reallocation of spectrum licences whose initial allocation was deemed inefficient (for example, because it was the outcome of a lottery) has been slow and costly, so much so that the U.S. Government decided to organize and run a centralized exchange, the recently concluded "incentive auction", in which revenue generation was an explicit part of the Government's objective.[3]

The theoretical literature on ex post efficient bilateral trade has identified private information as a source of often insurmountable transaction costs. Relaxing the assump-

[1]For example, Bleakley and Ferrie (2014) show that initial land parcel size after the opening of the frontier in Georgia predicts farm size essentially one-for-one for 50-80 years after land opening, with the effect of initial conditions attenuating gradually and disappearing only after 150 years. Milgrom (2004) makes a similar point in the context of the allocation of radio spectrum licenses, and Che and Cho (2011) describe vividly the inefficiencies associated with the Oklahoma land rush at the turn to the 20th century. Interestingly, Coase's own argument (Coase, 1959) favoring the use of auctions to allocate spectrum licenses is consistent with the notion that subsequent market transactions will not easily fix initial misallocations, which is the central premise of the insightful Theorem that bears his name (Coase, 1960) and that continues to be influential in public policy debates. As a case in point, consider Fowlie and Perloff (2013), who write in their abstract: "Standard economic theory predicts that if property rights to pollute are clearly established, equilibrium outcomes in an efficient emissions permit market will be independent of how the emissions permits are initially distributed."

[2]For legal and other costs associated with business deadlock, see, for example, Brooks, Landeo, and Spier (2010), Landeo and Spier (2014b) and Landeo and Spier (2014a) and the accounts therein. Bleakley and Ferrie (2014) document the time-consuming nature of reallocating land property that was initially allocated using lotteries. Popular press reports tell of the challenges modern China faces in reallocating farms lots that are deemed inefficiently small; see, for example, https://www.ft.com/content/9d18ee2a-a1a7-11e6-86d5-4e36b35c3550. Milgrom (2004) provides an account of the slow reallocation of spectrum licenses.

[3]For examples, see Milgrom and Segal (2015), https://www.fcc.gov/about-fcc/fcc-initiatives/incentive-auctions, and Milgrom (2017). For a recent review of the arguments and literature on two-sided exchanges, see, for example, Loertscher, Marx, and Wilkening (2015).

tion of extreme ownership structure underlying the bilateral trade setup, the partnership literature has highlighted that with appropriately chosen ownership structures and private values ex post efficient dissolution – that is, efficient reallocation of property rights – is possible.[4] While ex post efficiency is important and appealing, it is not necessarily the most appropriate assumption for all real-world applications. For example, designers may be interested in extracting revenue or there may be other resource costs associated with reallocating ownership shares. Theoretically, ex post efficiency may not be achievable without running a deficit because the ownership structure does not permit it or because interdependent values, which, for example, loom large in business deadlocks, may make efficient dissolution impossible for *any* ownership structure.[5] Among other things, this raises the question of what are the optimal mechanisms when the designer faces a binding revenue constraint.

Although the label may suggest otherwise, partnership models also offer a general framework to analyze markets with privately informed traders, incorporating as special cases extreme ownership models with buyers (that is, traders whose initial shares are zero) and sellers (that is, traders whose initial shares are equal to their maximum demand) and models with "interior" initial endowments, in which traders' positions – buy, sell, hold – are determined endogenously. Equivalently, the agents' shares can been reinterpreted as entitlements in a forward market, which are allocated before the resolution of any uncertainty.[6] To appreciate the richness of the partnership framework and its fundamental differences from standard mechanism design settings, note that the vast mechanism design literature has almost exclusively studied optimal mechanisms in settings with extreme ownership structures. For example, Myerson (1981) derives the revenue maximizing mechanism for the case with one seller who has full ownership. Likewise, Myerson and Satterthwaite (1983) derive the profit maximizing mechanism for a broker who intermediates between a privately informed buyer and a privately informed seller as well as the second-best mechanism for this broker. Gresik and Satterthwaite (1989) analyze convergence properties of the second-best mechanism as the market consisting of many buyers and sellers expands.[7] Little is known to date about the properties

[4]The formal literature on bilateral trade in the tradition of Myerson and Satterthwaite (1983) has its roots in Vickrey (1961) and Hurwicz (1972). Cramton, Gibbons, and Klemperer (1987) initiated the partnership literature.

[5]For analyses with private values assuming ex post efficiency, see, for example, Cramton, Gibbons, and Klemperer (1987), Che (2006), Schweizer (2006), and Figueroa and Skreta (2012). With interdependent values, Fieseler, Kittsteiner, and Moldovanu (2003) establish an impossibility result under ex post efficiency for any initial allocation of property rights.

[6]We thank Peter Cramton for bringing the forward market interpretation to our notice.

[7]Likewise, the closely related literature on double auctions, which includes Satterthwaite and Williams (1989), McAfee (1992), Rustichini, Satterthwaite, and Williams (1994), Tatur (2005) or Cripps

of the optimal mechanisms in markets in which the traders' positions are determined endogenously. This is a pervasive feature of all markets in which agents have some endowment of the asset being traded and determine, as a function of their value relative to those of the other traders, whether they want to buy, sell, or remain inactive. Being able to account for ex ante asymmetries among agents is a first-order issue here: For example, electricity generators using different forms of coal will exhibit systematic differences in their willingness to pay for pollution permits. Likewise, TV broadcasters differ systematically in their use of spectrum licenses from the one of telecom companies.

In this paper, we use the partnership framework to solve two economically significant problems. First and foremost, we derive the optimal, incentive compatible and individually rational mechanisms that maximize any convex combination of revenue and social surplus for any given initial ownership structure. For a fixed weight on revenue in the objective function and a given initial ownership structure, the *optimal dissolution mechanism* solves the problem that, for example, a court faces when asked to resolve a business deadlock or that a market designer faces who wants to defragment agricultural land whose disperse ownership consists of many inefficiently small lots. Equivalently, the optimal dissolution mechanism is the constrained efficient, or second-best, market mechanism when agents' trading positions – buy, sell, hold – are determined endogenously.[8] Second, using the optimal dissolution mechanisms, we solve for the *initially optimal ownership structure*. We distinguish between the case where the ownership structure is chosen to maximize the convex combination of revenue and social surplus and the case where the ownership structure maximizes social surplus net of a fixed cost that is borne if and only if dissolution occurs because both cases are relevant. For example, the allocation of pollution permits that are subsequently being traded in secondary markets or the determination of property rights (e.g. by the US Congress prior to the incentive auction or by courts prior to the resolution of legal disputes) correspond to the former. In contrast, business partners are more likely to choose ownership structures that maximize their joint expected surplus net of dissolution costs.

Methodologically, solving these problems poses major challenges, perhaps explaining the paucity of economists' understanding of these problems to date. From a methodological perspective, the key contribution of the paper is that it solves the first problem, not only because without this solution there is none to the second one, but primarily

and Swinkels (2006), also presumes extreme ownership.

[8] In this sense, the "incentive auction" was a special case of such a general market mechanism insofar as, for all intents and purposes, agents were either sellers or buyers. (It is true that the same entity could buy or sell; however, the goods on each side of the market were different – sellers sold broadcast licenses and buyers acquired telecom licenses.)

because of the intertwined nature of the problem. At the heart of the challenge for deriving the optimal dissolution mechanism lies the fact that partnerships with shared initial ownership create countervailing incentives (Lewis and Sappington, 1989): As an agent may end up buying additional shares or selling his share, his expected utility is typically minimized for types for whom the expected after-dissolution share equals the initial ownership share. These worst-off types do not get any information rent because they are indifferent between over- and underreporting. As they depend on the allocation rule, the worst-off types – which are the types for which the individual rationality constraint binds – are endogenous to the design problem. Note that this problem is absent from the standard mechanism design settings with extreme ownership because there an agent's worst-off type is known a priori – it is the lowest possible type if the agent is a buyer and the highest possible type if the agent is a seller.[9]

We overcome the problem of simultaneously determining the optimal allocation rule and the endogenous worst-off types by noticing and exploiting a saddle point property of the problem. Given a critical type for each agent, we define the virtual surplus as the value of the allocation in terms of virtual types. An agent's virtual type equals his virtual cost for types below the critical type and his virtual valuation for types above it, reflecting binding upward and downward incentive constraints. We then show that there is an essentially unique combination of critical types and an allocation rule such that, first, the allocation rule maximizes the virtual surplus given the critical types, and second, the critical types are worst-off types under the allocation rule. This is the allocation rule of all optimal dissolution mechanisms. Because virtual costs always exceed virtual valuations, the optimal dissolution mechanisms allocate based on ironed virtual type functions that are flat for types around the critical type. For some ownership structures, critical types are such that ties in terms of ironed virtual types happen with strictly positive probability. In this case, a suitably specified tie-breaking rule is an essential ingredient to the optimal allocation rule (ensuring that the critical type of each agent is worst off).

For the important special case of identical distributions and equal shares, we introduce a simple two-stage game that implements the optimal dissolution mechanism for a bilateral partnership. In this game, the designer first asks the agents to report "buy", "sell" or "hold". Trade occurs at posted prices that are contingent on the agents' reports

[9]The initial shares in our model represent type-dependent outside options. For the case of a single agent, principal-agent problems with this feature are well-understood (see Jullien, 2000, for the most general treatment). Much less is known for the case of multiple agents, making our analysis of such a setting a relevant contribution also to that literature.

unless both agents report "buy" or both report "sell".[10] If both report "buy" ("sell"), a standard (reverse) auction ensues to allocate the good efficiently.

The derivation of the optimal dissolution mechanism takes the initial ownership structure as given. In the real-world, however, ownership structures are often deliberately chosen. As mentioned, business partners choose their ownership structure. Likewise, governments allocate pollution permits, thereby determining the endowments different firms have before trading in secondary markets. Moreover, when there is legal uncertainty as to the exact nature of property rights, governments also play a role in determining initial ownership structures. This was, for example, the case when in the lead-up to the "incentive auction" the U.S. Congress determined that holders of an FCC spectrum license were entitled to use their licenses should they choose not to sell in the auction (Milgrom, 2017). Similar choices may be made by governments and courts prior to reallocating assets in countries with vaguely defined property rights. This raises the question as to what ownership structures should be chosen initially, that is, before the realization of private information. With the optimal dissolution mechanism at hand, we are able to answer this question. With identical distributions, equal initial ownership is optimal regardless of whether the ownership structure maximizes a convex combination of revenue and social surplus or social surplus net of the fixed cost of dissolution. Although with identical distributions equal ownership is optimal for any weight on revenue, the set of initially optimal ownership structures is larger when this weight is smaller, assuming dissolution is possible.[11] In contrast, with non-identical distributions, the ownership structure that maximizes social surplus net of a fixed cost of dissolution may not be part of the set of optimal ownership structure that maximize a convex combination of revenue and social surplus (and generate revenue equal to this fixed cost). Extreme initial ownership that, given the fixed cost, prohibits subsequent dissolution may be optimal in the former case even when interior initial ownership is optimal in the latter.

With notable exceptions, which we discuss below, the literature on partnership dissolution has focused on ex post efficient allocation rules and on the question under what conditions on distributions, valuations, and property rights ex post efficient reallocation is possible subject to incentive compatibility and individual rationality without

[10]To be precise, if both agents report "hold", there is no trade.

[11]This symmetric ownership structure is detail-free (Wilson, 1987) insofar as it does not depend on the specifics of the distribution, provided the distribution is the same for every agent. (The fact that the optimal ownership structure given identical distributions is generically not a singleton provides some robustness in a similar sense as in Neeman (1999)). However, the required dissolution mechanism need not be detail-free. For ex post efficiency and symmetric ownership, the $k+1$-price auction of Cramton, Gibbons, and Klemperer (1987) provides a detail-free mechanism. Whether this can be extended to revenue extraction and interdependent values is an open question. For detail-free dissolution mechanisms for asymmetric bilateral partnerships with private values, see Wasser (2013).

running a deficit. For the case in which all agents draw their types from the same distribution, Cramton, Gibbons, and Klemperer (1987) and Fieseler, Kittsteiner, and Moldovanu (2003) analyzed, respectively, models with private values and with interdependent values. Cramton, Gibbons, and Klemperer showed that with equal ownership, ex post efficiency is always possible. In contrast, Fieseler, Kittsteiner, and Moldovanu established that if interdependence is positive, ex post efficient reallocation may be impossible for any initial ownership structure. Their analysis gives thus additional salience to the question of what are optimal dissolution mechanisms, which is part of our study. Subsequent contributions with interdependent values were made by Kittsteiner (2003), Jehiel and Pauzner (2006), and Chien (2007). Considering symmetric bilateral partnerships, Kittsteiner (2003) performed a first attack on the problem of having to avoid deficits when valuations are positively interdependent. He showed that adding veto rights restores individual rationality of double-auctions but noticed at the same time that the resulting allocation is sub-optimal by providing a superior mechanism for an example with uniformly distributed types, which he (as we show) correctly conjectured to be the second-best mechanism. One contribution of our paper is that it generally derives the optimal mechanisms, thereby providing a benchmark to evaluate specific mechanisms that are or have been proposed to be used in practice, such as Kittsteiner's and the ones analyzed by Brooks, Landeo, and Spier (2010) and Landeo and Spier (2014b).

Focusing on private values, Che (2006) and Figueroa and Skreta (2012), with the latter building on the results of Schweizer (2006), extended the analysis to settings where each agent's type is drawn from a different distribution. When distributions can be ranked by stochastic dominance, Che and Figueroa and Skreta show that the ownership structure that maximizes revenue, given an ex post efficient allocation rule, assigns larger shares to stronger agents. Segal and Whinston (2011) provide, amongst other things, a generalization of the results of Schweizer (2006) to interdependent values. However, their conditions for possibility of ex post efficiency with interdependent values preclude those under which Fieseler, Kittsteiner, and Moldovanu (2003) establish impossibility.

To the best of our knowledge, the following are the only papers that analyze objectives other than ex post efficiency for partnership models with multilateral private information. Segal and Whinston (2016) study a second-best bargaining problem under a liability rule with two agents and private values. Our work complements theirs. While Segal and Whinston study a richer class of property rights, called liability rules, their analysis in this part of the paper is confined to two agents, private values, and the second-best mechanism, taking as given the initial allocation of property rights. In contrast, we first characterize the efficient frontier for an arbitrary number of agents, allowing for interdependent values and asymmetric distributions, and then derive the optimal ownership structure for any such partnership. Mylovanov and Tröger (2014) solve the informed principal problem one obtains when maximizing one agent's payoff in a bilateral partnership with private values. Our analysis differs from theirs insofar as our designer is not a member of the partnership and his objective attaches the same welfare weight to all agents. Other precursors to our paper are Lu and Robert (2001) and the unpublished paper by Chien (2007). Lu and Robert study the same objective function as we do in the derivation of optimal dissolution mechanisms but they confine attention to private values and identical type distributions, and they do not address which allocation of initial shares is optimal. Allowing for interdependent values, Chien solves for the second-best mechanism under given initial ownership for the special case of two agents. Our approach is both simpler and more general. Moreover, unless types are identically distributed, the second-best mechanism differs from what Chien's analysis suggests.

The remainder of this paper is organized as follows. Section 2 introduces the setup as well as basic mechanism design results. Section 3 derives and characterizes the optimal dissolution mechanisms, discusses second-best mechanisms, and introduces a simple implementation game. Section 4 determines the optimal initial ownership structures. Section 5 concludes. The Appendix contains omitted proofs.

2 Model

2.1 Setup

There is a set of n risk-neutral agents $\mathcal{N} := \{1, 2, \ldots, n\}$ who jointly own one object. Each agent $i \in \mathcal{N}$ owns share $r_i \in [0, 1]$ in the object, where $\sum_{i \in \mathcal{N}} r_i = 1$. Accordingly, the initial property rights are represented by a point $\mathbf{r} := (r_1, \ldots, r_n)$ in the $(n-1)$-dimensional standard simplex $\Delta^{n-1} := \left\{\mathbf{r} \in [0,1]^n : \sum_{i=1}^n r_i = 1\right\}$.

Each agent i privately learns his type x_i, which is a realization of the continuous random variable X_i. Each X_i is independently distributed according to a twice continuously differentiable cumulative distribution function F_i with support $[0, 1]$ and density f_i. Agent i's ex post valuation for the object is

$$v_i(\mathbf{x}) := x_i + \sum_{j \neq i} \eta(x_j)$$

where $\mathbf{x} := (x_1, \ldots, x_n)$ and where η is a differentiable function with $\eta'(x_j) < 1$ for all x_j. Agent i's status-quo utility from owning share r_i is $r_i v_i(\mathbf{x})$.

Dissolving the partnership results in a reallocation of initial property rights $\mathbf{r}$ and monetary transfers. By the Revelation Principle, it is without loss to focus on incentive compatible direct dissolution mechanisms. A direct dissolution mechanism $(\mathbf{s}, \mathbf{t})$ consists of an allocation rule $\mathbf{s}\colon [0,1]^n \to \Delta^{n-1}$ and a payment rule $\mathbf{t}\colon [0,1]^n \to \mathbb{R}^n$, where $\mathbf{s}(\mathbf{x}) = \big(s_1(\mathbf{x}), \dots, s_n(\mathbf{x})\big)$ and $\mathbf{t}(\mathbf{x}) = \big(t_1(\mathbf{x}), \dots, t_n(\mathbf{x})\big)$. The agents report their types $\mathbf{x}$ whereupon agent i receives share $s_i(\mathbf{x})$ and pays the amount $t_i(\mathbf{x})$, resulting in ex post payoff $v_i(\mathbf{x})s_i(\mathbf{x}) - t_i(\mathbf{x})$.[12]

Define $S_i(x_i) := E[s_i(x_i, \mathbf{X}_{-i})]$ and $T_i(x_i) := E[t_i(x_i, \mathbf{X}_{-i})]$ to be the interim expected share and payment of agent i. Moreover, let

$$U_i(x_i) := E[v_i(x_i, \mathbf{X}_{-i})\,(s_i(x_i, \mathbf{X}_{-i}) - r_i)] - T_i(x_i)$$

denote i's interim expected net payoff from taking part in the dissolution. A direct dissolution mechanism is Bayesian incentive compatible if

$$U_i(x_i) \geq E[v_i(x_i, \mathbf{X}_{-i})\,(s_i(\tilde{x}_i, \mathbf{X}_{-i}) - r_i)] - T_i(\tilde{x}_i) \quad \forall x_i, \tilde{x}_i \in [0,1], i \in \mathcal{N} \tag{IC}$$

and interim individually rational if

$$U_i(x_i) \geq 0 \quad \forall x_i \in [0,1], i \in \mathcal{N}. \tag{IR}$$

The designer's objective is to maximize a weighted sum of the ex ante expected social surplus $E\Big[\sum_i v_i(\mathbf{X})s_i(\mathbf{X})\Big]$, which is the value of the final allocation, and the ex ante expected revenue $E\Big[\sum_i t_i(\mathbf{X})\Big]$ subject to the incentive compatibility and individual rationality constraints. Suppose the designer puts weight $\alpha \in [0,1]$ on revenue and let

$$W_\alpha(\mathbf{s}, \mathbf{t}) := (1-\alpha) \sum_{i \in \mathcal{N}} E\Big[v_i(\mathbf{X})s_i(\mathbf{X})\Big] + \alpha \sum_{i \in \mathcal{N}} E\Big[t_i(\mathbf{X})\Big].$$

In Section 3, we will take the initial property rights $\mathbf{r}$ as given and study *optimal dissolution mechanisms* that solve

$$\max_{\mathbf{s}, \mathbf{t}} \; W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR).} \tag{1}$$

Note that the initial shares $\mathbf{r}$ enter this problem solely through the constraint (IR).

[12] Note that restricting attention to deterministic allocation rules is without loss of generality since payoffs are linear in the ex post shares. Agent i obtaining share $s_i = \sigma$ can equivalently be interpreted as i becoming the sole owner with probability σ and some other agent becoming the sole owner with probability $1 - \sigma$.

Optimal dissolution mechanisms will be denoted by $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$.

In Section 4, we will then turn to analyzing *optimal ownership structures* $\mathbf{r}^*$ that solve

$$\max_{\mathbf{r}} W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}}) \quad = \quad \max_{\mathbf{r},\mathbf{s},\mathbf{t}} W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR).} \tag{2}$$

2.2 Incentive compatibility, worst-off types, and virtual surplus

The standard characterization of Bayesian incentive compatibility applies to our environment (see, e.g., Myerson, 1981): (IC) holds if and only if

$$S_i \text{ is nondecreasing,} \tag{IC1}$$

$$U_i(x_i) = U_i(\hat{x}_i) + \int_{\hat{x}_i}^{x_i} (S_i(z) - r_i)dz \quad \forall x_i, \hat{x}_i \in [0,1]. \tag{IC2}$$

For a given monotone allocation rule, payoff equivalence (IC2) pins down interim expected payoffs U_i and payments T_i up to a constant.

Consider a dissolution mechanism $(\mathbf{s}, \mathbf{t})$ that satisfies (IC1) and (IC2). Let the set of *worst-off types* of agent i be denoted by $\Omega_i(\mathbf{s}) := \arg\min_{x_i} U_i(x_i)$. By (IC2), U_i is differentiable almost everywhere and $U_i'(x_i) = S_i(x_i) - r_i$ wherever U_i is differentiable. The monotonicity of S_i implies the following characterization of the set of worst-off types (see also Cramton, Gibbons, and Klemperer, 1987, Lemma 2). If there is an x_i such that $S_i(x_i) = r_i$, then $\Omega_i(\mathbf{s})$ is a (possibly degenerate) interval and

$$\Omega_i(\mathbf{s}) = \{\, x_i : S_i(x_i) = r_i \,\}.$$

If $S_i(x_i) \neq r_i$ for all $x_i \in [0,1]$, then $\Omega_i(\mathbf{s})$ is a singleton and

$$\Omega_i(\mathbf{s}) = \{\, x_i : S_i(z) < r_i \; \forall z < x_i \text{ and } S_i(z) > r_i \; \forall z > x_i \}.$$

Let $\Omega(\mathbf{s}) := \Omega_1(\mathbf{s}) \times \cdots \times \Omega_n(\mathbf{s})$.

In addition to identifying the set of worst-off types, (IC2) also allows us to eliminate $\mathbf{t}$ from the designer's objective and rewrite it as a function of the the interim payoff of an arbitrarily fixed critical type for each agent and the virtual surplus generated by $\mathbf{s}$ under these critical types. To do so, we first define, for each i, the α-weighted virtual cost and virtual valuation

$$\psi^S_{\alpha,i}(x_i) := x_i - \eta(x_i) + \alpha \frac{F_i(x_i)}{f_i(x_i)} \quad \text{and} \quad \psi^B_{\alpha,i}(x_i) := x_i - \eta(x_i) - \alpha \frac{1 - F_i(x_i)}{f_i(x_i)}.$$

The first part of $\psi^S_{\alpha,i}$ and $\psi^B_{\alpha,i}$, the term $x_i - \eta(x_i)$, represents the effect of i's type on the cost from reducing and gain from increasing, respectively, i's share.[13] The second part, with the designer's revenue weight α, accounts for the information rent that has to be granted to i to prevent him from overstating and understating, respectively, his type.

For an exogenously fixed *critical type* $\hat{x}_i \in [0,1]$, we define agent i's *virtual type* function given $\hat{x}_i$ as

$$\psi_{\alpha,i}(x_i, \hat{x}_i) := \begin{cases} \psi^S_{\alpha,i}(x_i) & \text{if } x_i < \hat{x}_i, \\ \psi^B_{\alpha,i}(x_i) & \text{if } x_i > \hat{x}_i. \end{cases}$$

The *virtual surplus* under allocation rule $\mathbf{s}$ and exogenously fixed critical types $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_n)$ is then given by

$$\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) := E\Bigg[\sum_{i \in \mathcal{N}} (s_i(\mathbf{X}) - r_i)\psi_{\alpha,i}(X_i, \hat{x}_i)\Bigg],$$

i.e., the expected gains from trade in terms of virtual types given $\hat{\mathbf{x}}$ when reallocating property rights from $\mathbf{r}$ to $\mathbf{s}$. Using standard techniques, we obtain the following lemma.

Lemma 1. *Suppose the dissolution mechanism* $(\mathbf{s}, \mathbf{t})$ *satisfies* (IC1) *and* (IC2)*. Then,*

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) - \alpha \sum_{i \in \mathcal{N}} U_i(\hat{x}_i) + (1-\alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) r_i] \qquad \textit{for all } \hat{\mathbf{x}} \in [0,1]^n. \quad (3)$$

Moreover,

$$\Omega(\mathbf{s}) = \underset{\hat{\mathbf{x}} \in [0,1]^n}{\arg\min}\, \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \quad (4)$$

Proof. See Appendix B.1. □

According to (3), we can write the designer's objective for any exogenously fixed critical types $\hat{\mathbf{x}}$ as the virtual surplus given $\hat{\mathbf{x}}$ minus the α-weighted sum of the interim payoffs of the critical types plus $(1-\alpha)$ times the value of the initial allocation. Since for a given allocation rule $\mathbf{s}$, (3) is constant over all $\hat{\mathbf{x}}$, the critical types with the smallest interim payoffs must also be the critical types that minimize the virtual surplus, implying (4). Identifying the worst-off types as the critical types that minimize the virtual surplus will prove useful below.

[13] The change in surplus from moving the object from j to i is $v_i(\mathbf{x}) - v_j(\mathbf{x}) = x_i - \eta(x_i) - x_j + \eta(x_j)$.

2.3 Regularity and virtual distributions

We will throughout impose the regularity assumption that each agent's α-weighted virtual cost and valuation is strictly increasing, i.e.,

$$\frac{d}{dx_i}\psi^S_{\alpha,i}(x_i) > 0 \quad \text{and} \quad \frac{d}{dx_i}\psi^B_{\alpha,i}(x_i) > 0 \quad \text{for all } x_i \in [0,1] \text{ and } i \in \mathcal{N}. \tag{5}$$

This represents a joint assumption on α, η, and $F_1, \ldots, F_n$. Note that the higher α and $\eta'(\cdot)$ are, the more restrictive is (5) for F_i. (5) is satisfied for all α and η if each f_i is log-concave.

For our analysis below, it is useful to define the cumulative distribution functions $G^S_{\alpha,i}$ and $G^B_{\alpha,i}$ of agent i's virtual cost $\psi^S_{\alpha,i}(X_i)$ and virtual valuation $\psi^B_{\alpha,i}(X_i)$: Under (5), we have

$$G^K_{\alpha,i}(y) := \begin{cases} 0 & \text{if } y < \psi^K_{\alpha,i}(0), \\ F_i\Big((\psi^K_{\alpha,i})^{-1}(y)\Big) & \text{if } y \in \Big[\psi^K_{\alpha,i}(0), \psi^K_{\alpha,i}(1)\Big], \\ 1 & \text{if } y > \psi^K_{\alpha,i}(1) \end{cases}$$

for $K \in \{S, B\}$ and $i \in \mathcal{N}$. Observe that for every i and y, $G^S_{\alpha,i}(y) \leq F_i(y) \leq G^B_{\alpha,i}(y)$.

3 Optimal Dissolution Mechanisms

3.1 General Partnerships

In the following, we will determine the solution to the designer's problem stated in (1). From Subsection 2.2 follows that we can replace the constraints (IC) and (IR) with (IC1), (IC2), and $U_i(\omega_i) \geq 0$ for all i and $\omega_i \in \Omega_i(\mathbf{s})$. Define $\mathfrak{S} := \Big\{\mathbf{s} : S_i \text{ is nondecreasing for each } i \in \mathcal{N}\Big\}$. Consequently, (IC1) is equivalent to $\mathbf{s} \in \mathfrak{S}$.

Consider an allocation rule $\mathbf{s} \in \mathfrak{S}$ and some worst-off types $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \Omega(\mathbf{s})$. Under (IC2), (3) in Lemma 1 implies that we can write the designer's objective as

$$\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) - \alpha \sum_{i \in \mathcal{N}} U_i(\omega_i) + (1-\alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) r_i]. \tag{6}$$

Note that the individual rationality constraint $U_i(\omega_i) \geq 0$ is binding when choosing payments $\mathbf{t}$ that maximize (6) for a given $\mathbf{s}$. $U_i(\omega_i) = 0$ and (IC2) imply that any optimal $\mathbf{t}$ has to be such that interim expected payments satisfy, for all i,

$$T_i(x_i) = E[v_i(x_i, \mathbf{X}_{-i})\,(s_i(x_i, \mathbf{X}_{-i}) - r_i)] - \int_{\omega_i}^{x_i} (S_i(z) - r_i) dz. \tag{7}$$

It remains to determine the optimal allocation rule. Since the second term in the objective (6) is zero under optimal payments and the third term is independent of the dissolution mechanism, we can restrict attention to maximizing $\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) = \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$, where the equality follows from (4) in Lemma 1. Consequently, an optimal allocation rule $\mathbf{s}^\mathbf{r}$ has to satisfy

$$\mathbf{s}^\mathbf{r} \in \arg\max_{\mathbf{s}\in\mathfrak{S}} \min_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \tag{8}$$

Instead of directly solving the max-min problem (8), we will look for a *saddle point* $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ of $\widetilde{W}_\alpha$ that satisfies

$$\mathbf{s}^* \in \arg\max_{\mathbf{s}\in\mathfrak{S}} \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*), \tag{9}$$

$$\boldsymbol{\omega}^* \in \arg\min_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}^*, \hat{\mathbf{x}}). \tag{10}$$

For a saddle point, (9) requires that the allocation rule $\mathbf{s}^*$ maximizes the virtual surplus $\widetilde{W}_\alpha$ under given critical types $\boldsymbol{\omega}^*$ whereas (10) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{s}^*$, i.e., $\boldsymbol{\omega}^* \in \Omega(\mathbf{s}^*)$.

Note that if a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists, then $\mathbf{s}^\mathbf{r}$ solves the problem in (8) if and only if $(\mathbf{s}^\mathbf{r}, \boldsymbol{\omega}^*)$ is a saddle point.[14] In the following, we will show that a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists and that $\mathbf{s}^*$ is essentially unique.[15] The characterization of optimal dissolution mechanisms we will thereby obtain represents the main result of this section. We will proceed by first determining the class of allocation rules that is consistent with (9). Then we will argue that an essentially unique member of this class also satisfies (10).

Consider the optimization problem in (9). Pointwise maximization of

$$\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) = E\Big[\sum\nolimits_{i\in\mathcal{N}} (s_i(\mathbf{X}) - r_i)\psi_{\alpha,i}(X_i, \omega_i^*)\Big]$$

would require allocating the object to the agent i with the highest virtual type $\psi_{\alpha,i}(x_i, \omega_i^*)$. Yet, since $\psi^S_{\alpha,i}(x_i) > \psi^B_{\alpha,i}(x_i)$ for all x_i, $\psi_{\alpha,i}(x_i, \omega_i^*)$ is not monotone at ω_i^*, resulting in the monotonicity constraint $\mathbf{s} \in \mathfrak{S}$ to be violated. The solution to (9) therefore involves ironing (Myerson, 1981): the object is allocated to an agent i with the highest *ironed*

[14] Suppose $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ satisfies (9) and (10). Then, $\min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}^*, \hat{\mathbf{x}}) = \widetilde{W}_\alpha(\mathbf{s}^*, \boldsymbol{\omega}^*) \geq \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) \geq \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$ for all $\mathbf{s} \in \mathfrak{S}$ and hence $\mathbf{s}^*$ solves the problem in (8). Conversely, for all $\mathbf{s}^\mathbf{r}$ that satisfy (8), the above has to hold with equality, implying that $(\mathbf{s}^\mathbf{r}, \boldsymbol{\omega}^*)$ is a saddle point.

[15] $\mathbf{s}^*$ is unique up to the exact specification of a tie-breaking rule.

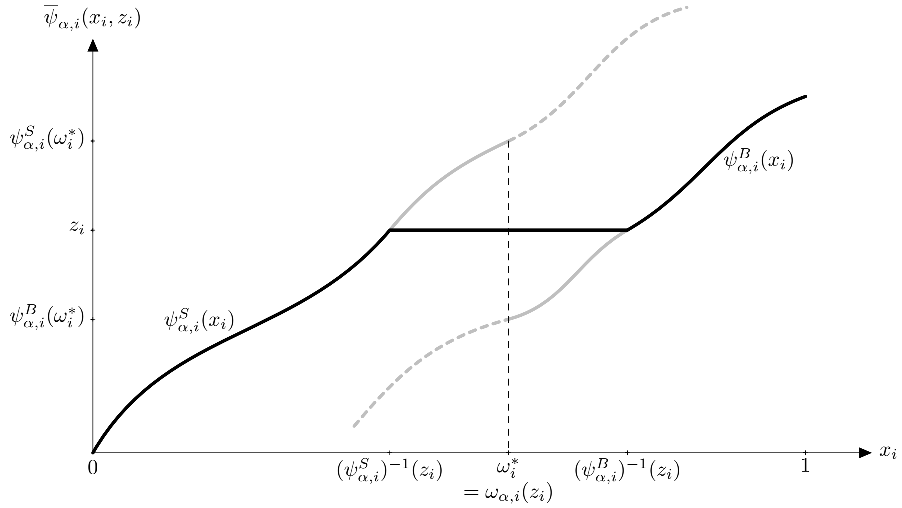


Figure 1: Ironed virtual type function.

virtual type

$$\overline{\psi}_{\alpha,i}(x_i, z_i) := \begin{cases} \psi^S_{\alpha,i}(x_i) & \text{if } \psi^S_{\alpha,i}(x_i) < z_i, \\ z_i & \text{if } \psi^B_{\alpha,i}(x_i) \leq z_i \leq \psi^S_{\alpha,i}(x_i), \\ \psi^B_{\alpha,i}(x_i) & \text{if } z_i < \psi^B_{\alpha,i}(x_i) \end{cases}$$

where the ironing parameter $z_i \in \left[\psi^B_{\alpha,i}(\omega^*_i), \psi^S_{\alpha,i}(\omega^*_i)\right]$ is the unique solution to

$$E\left[\psi_{\alpha,i}(X_i, \omega^*_i)\right] = E\left[\overline{\psi}_{\alpha,i}(X_i, z_i)\right]. \tag{11}$$

According to (11), there is a one-to-one relation between the critical type ω^*_i and the corresponding ironing parameter z_i, which can be expressed in closed form as follows. It is straightforward to verify that $\frac{d}{d\omega^*_i} E\left[\psi_{\alpha,i}(X_i, \omega^*_i)\right] = \alpha$ and that $\frac{d}{dz_i} E\left[\overline{\psi}_{\alpha,i}(X_i, z_i)\right] = G^B_{\alpha,i}(z_i) - G^S_{\alpha,i}(z_i) > 0$. Moreover, note that for $\omega^*_i = 0$ and $\omega^*_i = 1$, (11) yields $z_i = \psi^B_{\alpha,i}(0)$ and $z_i = \psi^S_{\alpha,i}(1)$, respectively. Using implicit differentiation, we can solve (11) for ω^*_i, resulting in

$$\omega^*_i = \omega_{\alpha,i}(z_i) := \frac{1}{\alpha} \int_{\psi^B_{\alpha,i}(0)}^{z_i} \left(G^B_{\alpha,i}(y) - G^S_{\alpha,i}(y)\right) dy. \tag{12}$$

Note that $\omega_{\alpha,i}(\cdot)$ is a continuous and strictly increasing function.

Figure 1 illustrates the ironed virtual type function. Agent i's ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i)$ is constant and equal to z_i for an interval of types that contains the critical type $\omega_{\alpha,i}(z_i)$ and it is strictly increasing otherwise. Any allocation rule consistent with (9) allocates based on ironed virtual types and hence features for each agent bunching around the critical type. Note that (9) does not pin down the allocation when several agents tie for the highest ironed virtual type. To handle this indeterminacy, we next introduce a convenient class of tie-breaking rules.

Let H denote the set of all $n!$ permutations $(h(1), h(2), \ldots, h(n))$ of $(1, 2, \ldots, n)$. We will call each $h \in H$ a hierarchy among the agents in $\mathcal{N}$. A *hierarchical tie-breaking rule* breaks ties in favor of the agent who is the highest in the hierarchy: If the set of agents $\mathcal{I} \subseteq \mathcal{N}$ tie for the highest ironed virtual type and there is hierarchical tie-breaking according to hierarchy h, the object is assigned to agent $\arg\max_{i \in \mathcal{I}} h(i)$. Under a *split hierarchical tie-breaking rule* $\mathbf{a}$, ownership of the object is split up into $n!$ shares $\mathbf{a} := (a_1, \ldots, a_{n!}) \in \Delta^{n!-1}$, one for each hierarchy in $H = \{h_1, \ldots, h_{n!}\}$, and then each a_l is assigned according to hierarchy h_l, i.e., to agent $\arg\max_{i \in \mathcal{I}} h_l(i)$.[16] The outcome in terms of interim expected shares $S_1, \ldots, S_n$ of any tie-breaking rule can equivalently be obtained by a split hierarchical tie-breaking rule $\mathbf{a}$.

Define the *ironed virtual type allocation rule* $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ with ironing parameters $\mathbf{z} = (z_1, \ldots, z_n)$ and split hierarchical tie-breaking rule $\mathbf{a}$ as, for all $i \in \mathcal{N}$,

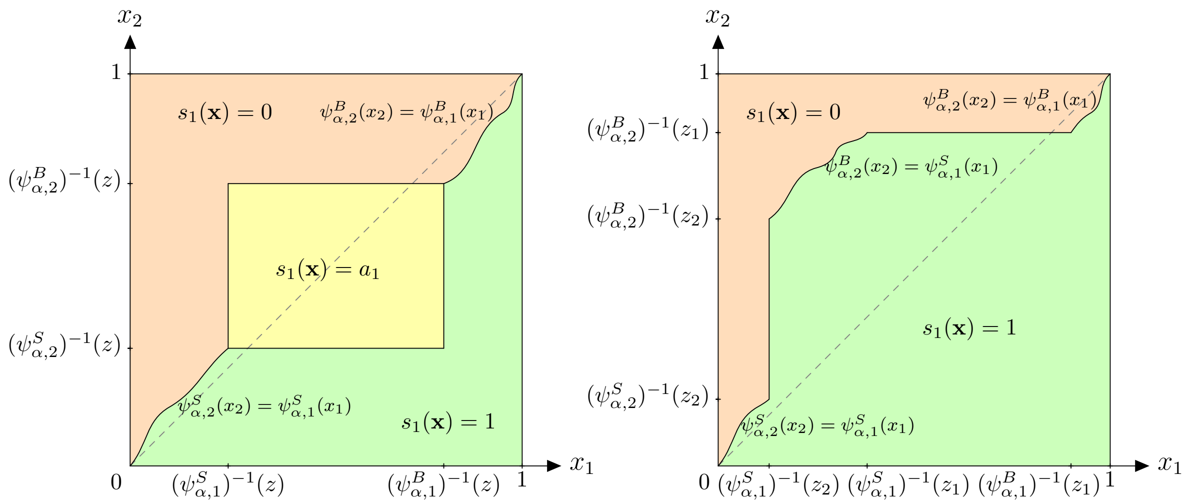
$$s_i^{\mathbf{z},\mathbf{a}}(\mathbf{x}) := \begin{cases} 1 & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) > \max_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j), \\ \sum_{h \in \hat{H}_i} a_h & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) = \max_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j), \\ 0 & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) < \max_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j), \end{cases}$$

where $\hat{H}_i := \left\{h \in H : h(i) > h(k) \ \forall k \in \arg\max_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j)\right\}$. For a given $\boldsymbol{\omega}^*$, $\mathbf{s}^* = \mathbf{s}^{\mathbf{z},\mathbf{a}}$ solves the problem in (9) for $\mathbf{z} = \left(\omega_{\alpha,1}^{-1}(\omega_1^*), \ldots, \omega_{\alpha,n}^{-1}(\omega_n^*)\right)$ and any tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$.

Figure 2 presents two examples of ironed virtual type allocation rules $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ for bilateral partnerships. The tie-breaking rule $\mathbf{a} = (a_1, a_2)$ is such that a_1 corresponds to the hierarchy according to which agent 1 beats agent 2. In both examples, the green (red) area represents all type realizations where agent 1's ironed virtual type is strictly greater (smaller) than agent 2's, resulting in full ownership of the object being allocated to agent 1 (2). In Panel (a) ironing parameters are set such that $z_1 = z_2 = z$ for some z. This implies that for all type realizations in the yellow area, the ironed virtual types are the same for both agents, which happens with strictly positive probability. In this case, share a_1 is allocated to agent 1 and share $a_2 = 1 - a_1$ to agent 2. The interim expected share of each agent's critical type $S_i(\omega_{\alpha,i}(z))$ therefore depends on the tie-breaking rule. In Panel (b) ironing parameters satisfy $z_1 > z_2$, implying that ties have probability zero and the tie-breaking rule does not affect interim expected shares.

Having established that all allocation rules consistent with (9) are equivalent to ironed

[16] An alternative interpretation is that one hierarchy h is randomly selected from H according to the probability distribution $\mathbf{a}$ over H and ties are then broken according to h.



(a) $z_1 = z_2 = z$ (b) $z_1 > z_2$

Figure 2: Ironed virtual type allocation rules for $n = 2$.

virtual type allocation rules $\mathbf{s}^{\mathbf{z},\mathbf{a}}$, we now turn to the second requirement for a saddle point. (10) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{s}^*$. A simultaneous solution to (9) and (10) hence corresponds to a vector of ironing parameters $\mathbf{z}$ and a tie-breaking rule $\mathbf{a}$ such that $\omega_{\alpha,i}(z_i) \in \Omega_i(\mathbf{s}^{\mathbf{z},\mathbf{a}})$ for each agent i. Note that because of the bunching property, the interim expected share $S_i^{\mathbf{z},\mathbf{a}}(x_i)$ is constant for an interval of types x_i that contains the critical type $\omega_{\alpha,i}(z_i)$. The characterization of the set of worst-off types in Subsection 2.2 hence implies that for critical types to be worst-off, we must have $S_i^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,i}(z_i)) = r_i$ for all $i \in \mathcal{N}$.

We will show that there is typically a unique $\mathbf{z}$ such that $S_i^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,i}(z_i)) = r_i$ for all i for some $\mathbf{a}$, yielding existence of a saddle point and a characterization of optimal dissolution mechanisms. To prove this result and make its statement precise, the following definitions will be useful. Let $\underline{z} := -\eta(0)$, $\overline{z} := 1 - \eta(1)$ and define the correspondence $\Gamma_n \colon [\underline{z}, \overline{z}]^n \to [0,1]^n$ such that

$$\Gamma_n(\mathbf{z}) := \Big\{ \Big(S_1^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,1}(z_1)), \ldots, S_n^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,n}(z_n)) \Big) : \mathbf{a} \in \Delta^{n!-1} \Big\}.$$

$\Gamma_n(\mathbf{z})$ yields the set of all vectors of expected shares for critical types $\omega_{\alpha,1}(z_1), \ldots, \omega_{\alpha,n}(z_n)$ that can be obtained with ironing parameters $\mathbf{z}$ and some tie-breaking rule $\mathbf{a}$. If $z_i = z_j$ for two agents i, j, there is a strictly positive probability for a tie and the expected shares depend on tie-breaking. $\Gamma_n(\mathbf{z})$ is singleton-valued if and only if $z_i \neq z_j$ for all i and $j \neq i$.

We are now ready to state our main result on optimal dissolution mechanisms.

Theorem 1. *For each* $\mathbf{r} \in \Delta^{n-1}$*, there exists a unique* $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ *such that* $\mathbf{r} \in \Gamma_n(\mathbf{z})$*. Let* $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$*. All optimal dissolution mechanisms* $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ *that solve* (1) *consist of*

- *an allocation rule* $\mathbf{s}^{\mathbf{r}}$ *that allocates the object to an agent* i *with the greatest ironed virtual type* $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$*, where ties are broken such that* $S_i^{\mathbf{r}}(\omega_{\alpha,i}(z_i^*)) = r_i$ *for all* $i \in \mathcal{N}$*,*
- *and a payment rule* $\mathbf{t}^{\mathbf{r}}$ *such that interim expected payments satisfy*

$$T_i^{\mathbf{r}}(x_i) = E[v_i(x_i, \mathbf{X}_{-i})\,(s_i^{\mathbf{r}}(x_i, \mathbf{X}_{-i}) - r_i)] - \int_{\omega_{\alpha,i}(z_i^*)}^{x_i} (S_i^{\mathbf{r}}(y) - r_i)dy \quad \text{for all } i \in \mathcal{N}.$$

A split hierarchical tie-breaking rule $\mathbf{a}^*$ *exists such that* $\mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ *is an optimal allocation rule.*

Proof. See Appendix A. □

The most challenging part of the proof is to establish the first line of Theorem 1, ensuring that the inverse correspondence $\Gamma_n^{-1}(\mathbf{r})$ is singleton-valued for all initial shares $\mathbf{r}$. In Appendix A, we uncover a recursive structure to Γ_n by partitioning its domain in a suitable way. This allows us to prove that Γ_n has the required properties by induction, using the tractable two-agent case as the base case. The existence of a unique $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ in turn implies that $\mathbf{s}^* = \mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ and $\boldsymbol{\omega}^* = \left(\omega_{\alpha,1}(z_1^*), \ldots, \omega_{\alpha,n}(z_n^*)\right)$ constitute a saddle point satisfying (9) and (10) for all split hierarchical tie-breaking rules $\mathbf{a}^*$ such that $\left(S_1^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,1}(z_1^*)), \ldots, S_n^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,n}(z_n^*))\right) = \mathbf{r}$. Consequently, $\mathbf{s}^{\mathbf{r}} = \mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ is an optimal allocation rule that solves the max-min problem (8). Any other optimal allocation rule $\mathbf{s}^{\mathbf{r}}$ may differ from $\mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ only with respect to the tie-breaking rule. Finally, interim expected payments are pinned down by payoff equivalence (IC2), as stated in (7).

For all tie-breaking rules $\mathbf{a}$, $\left(S_1^{\mathbf{z}^*,\mathbf{a}}(\omega_{\alpha,1}(z_1^*)), \ldots, S_n^{\mathbf{z}^*,\mathbf{a}}(\omega_{\alpha,n}(z_n^*))\right)$ is equal to the convex combination with coefficients $\mathbf{a}$ of the $n!$ vectors of the critical types' expected shares under hierarchical tie-breaking. $\Gamma_n(\mathbf{z}^*)$ is hence the convex hull of the expected shares under hierarchical tie-breaking. Once the ironing parameters $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ have been determined, finding a corresponding tie-breaking rule $\mathbf{a}^*$ to implement an optimal allocation rule is straightforward: one just has to find a convex combination of the extreme points of the convex hull $\Gamma(\mathbf{z}^*)$ that is equal to $\mathbf{r}$. Although according to our general definition tie-breaking rules $\mathbf{a}$ correspond to splitting ownership into $n!$ shares that are each allocated according to a different hierarchy, the optimal allocation rule can be implemented with a tie-breaking rule $\mathbf{a}^*$ that splits ownership into at most n shares (i.e., $a_h^* = 0$ for at least $n! - n$ hierarchies $h \in H$).[17]

[17] As argued above, $\mathbf{a}^*$ has to satisfy $\left(S_1^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,1}(z_1^*)), \ldots, S_n^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,n}(z_n^*))\right) = \mathbf{r}$, where the LHS is a

To summarize, optimal dissolution mechanisms reallocate shares based on ironed virtual types, where for each ownership structure, the intervals of types where bunching occurs are uniquely determined to ensure together with the tie-breaking rule that the corresponding critical type of each agent expects to retain his initial ownership share.

3.2 Bilateral Partnerships

To illustrate the working of the optimal dissolution mechanisms, we now specialize the setup to one with two agents. According to Theorem 1, an optimal dissolution mechanism allocates the object to the agent i with the higher ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$, where $(z_1^*, z_2^*) = \Gamma_2^{-1}(r_1, r_2)$. For bilateral partnerships, characterizing (z_1^*, z_2^*) further is possible at little additional cost.

Suppose $z_1^* > z_2^*$. Then, the critical type of agent 1 expects to obtain the object with probability $S_1(\omega_{\alpha,1}(z_1^*)) = G^B_{\alpha,2}(z_1^*)$ whereas the critical type of agent 2 expects to obtain the object with probability $S_2(\omega_{\alpha,2}(z_2^*)) = G^S_{\alpha,1}(z_2^*)$.[18] Moreover, these probabilities are equal to the initial shares r_1 and $r_2 = 1 - r_1$, making the critical types worst-off types. Consequently, all initial shares that are consistent with $z_1^* > z_2^*$ satisfy $(G^B_{\alpha,2})^{-1}(r_1) > (G^S_{\alpha,1})^{-1}(r_2)$. This is true for all $r_1 \in (\overline{r}_1, 1]$, where $\overline{r}_1$ uniquely solves $(G^B_{\alpha,2})^{-1}(\overline{r}_1) = (G^S_{\alpha,1})^{-1}(1 - \overline{r}_1)$.

Similarly, we find that $z_1^* < z_2^*$ if and only if $r_1 \in [0, \underline{r}_1)$, where $\underline{r}_1$ is the unique solution to $(G^S_{\alpha,2})^{-1}(\underline{r}_1) = (G^B_{\alpha,1})^{-1}(1 - \underline{r}_1)$. Observe that $0 < \underline{r}_1 < \overline{r}_1 < 1$ for all $\alpha > 0$ and that $\underline{r}_1$ is decreasing and $\overline{r}_1$ is increasing in α.

It follows that for $r_1 \in [\underline{r}_1, \overline{r}_1]$ we must have $z_1^* = z_2^*$. In this case agents tie for the highest ironed virtual type with positive probability. If agent i obtains share a_i in case of a tie, then i's critical type expects to obtain share $S_i(\omega_{\alpha,i}(z_i^*)) = a_i G^B_{\alpha,j}(z_i^*) + (1 - a_i) G^S_{\alpha,j}(z_i^*)$. The optimal allocation rule makes sure that this expected share is equal to r_i. We thus obtain the following corollary to Theorem 1.

Corollary 1. *Suppose $n = 2$. The optimal allocation rule $\mathbf{s}^{\mathbf{r}}$ allocates the object to the agent i who has the higher ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$ and in case of a tie assigns share a_1^* to agent 1 and $1 - a_1^*$ to agent 2.*

(i) If $r_1 \in [0, \underline{r}_1)$, then $z_1^* = (G^S_{\alpha,2})^{-1}(r_1) < (G^B_{\alpha,1})^{-1}(r_2) = z_2^*$ and $a_1^* \in [0, 1]$.

point in the convex hull of a set P of $n!$ points that represent the critical types' expected shares under hierarchical tie-breaking. As we show in the proof of Theorem 1, P lies in an $(n-1)$-dimensional hyperplane. By Carathéodory's theorem, there is an $\mathbf{a}^*$ that combines only n points out of P.

[18] To see this, note that the cumulative distribution function of agent i's ironed virtual type $Y_i = \overline{\psi}_{\alpha,i}(X_i, z_i^*)$ corresponds to $G^S_{\alpha,i}(y_i)$ for $y_i \le z_i^*$ and $G^B_{\alpha,i}(y_i)$ for $y_i > z_i^*$.

(ii) If $r_1 \in [\underline{r}_1, \overline{r}_1]$*, then* $z_1^* = z_2^* = z^*$*, where* z^* *and* a_1^* *are the unique solution to*

$$a_1^* G_{\alpha,2}^B(z^*) + (1 - a_1^*) G_{\alpha,2}^S(z^*) = r_1, \qquad a_1^* G_{\alpha,1}^S(z^*) + (1 - a_1^*) G_{\alpha,1}^B(z^*) = r_2.$$

(iii) If $r_1 \in (\overline{r}_1, 1]$*, then* $z_1^* = (G_{\alpha,2}^B)^{-1}(r_1) > (G_{\alpha,1}^S)^{-1}(r_2) = z_2^*$ *and* $a_1^* \in [0,1]$*.*

In cases (i) and (iii) of Corollary 1, ties occur with probability zero, which explains why ties can be broken arbitrarily, i.e., why any $a_1^* \in [0,1]$ is optimal. In contrast, for case (ii) the tie-breaking rule a_1^* of the optimal allocation rule is unique.

The optimal allocation rule described in Corollary 1 is illustrated in Figure 2. Panel (a) depicts case (ii) of Corollary 1 and Panel (b) case (iii), which after interchanging the agents' names also applies to case (i). The figures are drawn for a situation where $F_1 \neq F_2$, i.e., where agents draw their types from different distributions. From the figures we can infer how the optimal allocation rule for $\alpha > 0$ differs from the ex post efficient allocation rule that assigns the object to agent 1 (2) if (x_1, x_2) is below (above) the dashed 45-degree line.

Suppose the ownership structure is sufficiently symmetric such that $r_1 \in (\underline{r}_1, \overline{r}_1)$, implying optimal ironing parameters $z_1^* = z_2^* = z^*$. This corresponds to Panel (a) of Figure 2. Types $x_1 \in \left[(\psi_{\alpha,1}^S)^{-1}(z^*), (\psi_{\alpha,1}^B)^{-1}(z^*)\right]$ of agent 1 and types $x_2 \in \left[(\psi_{\alpha,2}^S)^{-1}(z^*), (\psi_{\alpha,2}^B)^{-1}(z^*)\right]$ of agent 2 all have the same ironed virtual type z^*. If both type realizations are within these intervals, share $a_1^* \in (0,1)$ of the object is assigned to agent 1, as represented by the yellow rectangle. This inefficiency of the allocation is reminiscent of the traditional under-supply by a monopolist and of auctions with revenue-maximizing reserve prices. If both agents draw a sufficiently high type, the object is allocated to the agent with the highest virtual valuation, whereas for sufficiently low types the allocation is based on comparing virtual costs. Thus the object may end up in the hands of the agent who values it less, resulting in a second kind of inefficiency, like in the optimal auction of Myerson (1981) with asymmetric bidders. Whereas the first kind of inefficiency is always present for $\alpha > 0$, the second kind vanishes if the agents' types are identically distributed.

As we increase r_1 within $[\underline{r}_1, \overline{r}_1]$, the share a_1^* increases and z^* may change (it stays constant if $F_1 = F_2$), until we reach $\overline{r}_1$ where $a_1^* = 1$. At this point, we leave the case underlying Panel (a) of Figure 2 and switch to the situation depicted in Panel (b). As we increase r_1 further, z_1^* increases and z_2^* decreases, eventually reaching $\overline{z}$ and $\underline{z}$, respectively, when $r_1 = 1$.

Now, consider $r_1 \in (\overline{r}_1, 1]$, which implies $z_1^* > z_2^*$ as in Panel (b) of Figure 2. If types $(x_1, x_2) \in \left[(\psi_{\alpha,1}^S)^{-1}(z_2^*), 1\right] \times \left[0, (\psi_{\alpha,2}^B)^{-1}(z_1^*)\right]$ realize, the optimal allocation rule assigns the object to agent 1 if his virtual cost $\psi^S_{\alpha,1}(x_1)$ is higher than the virtual valuation $\psi^B_{\alpha,2}(x_2)$ of agent 2. Otherwise, the object is assigned to agent 2. For type realizations within this region, the optimal allocation thus corresponds exactly to the allocation rules derived by Myerson and Satterthwaite (1983), giving rise to the same inefficiency. If $x_1 < (\psi^S_{\alpha,1})^{-1}(z^*_2)$, the object is allocated on the basis of virtual costs whereas if $x_2 > (\psi^B_{\alpha,2})^{-1}(z^*_1)$, the object is assigned to the agent with the higher virtual valuation. In those cases, we obtain again the second kind of inefficiency that disappears if types are drawn from the same distribution. Note that for $r_1 = 1$, where $(\psi^S_{\alpha,1})^{-1}(z^*_2) = 0$ and $(\psi^B_{\alpha,2})^{-1}(z^*_1) = 1$, the optimal allocation rule coincides with the solution of Myerson and Satterthwaite (1983) on the entire type space $[0,1]^2$. This is, of course, consistent with the partnership model approaching a bilateral trade setting where agent 1 is the seller and agent 2 the buyer as r_1 approaches 1.

As α increases while r_1 is kept fixed, the inefficiency of the optimal allocation increases: In Panel (a) the yellow rectangle with tie-breaking becomes larger and in Panel (b) the demarcation line where 1's virtual cost coincides with 2's virtual valuation moves upward and to the left. This is because a higher α makes the difference between virtual types and actual net types $x_i - \eta(x_i)$ larger. The comparative static effects of increasing the (positive) interdependence of valuations on the optimal allocation are similar to the effects of increasing α under private values. This is easiest to see for the case with linear interdependence $\eta(x) = ex$ with $e < 1$. In this case, i's virtual type $\psi^K_{\alpha,i}(x_i)$ is larger than j's virtual type $\psi^L_{\alpha,j}(x_j)$ with $K, L \in \{B, S\}$ if and only if for private values (i.e., $\eta'(x) = 0$) $\psi^K_{\alpha/(1-e),i}(x_i) \geq \psi^L_{\alpha/(1-e),j}(x_j)$. The effect of increasing e in the model with linear interdependence will thus be qualitatively the same as increasing α in the private values model.

3.3 Maximizing Surplus under a Revenue Constraint

Having determined the optimal dissolution mechanisms that maximizes W_α for any $\alpha \in [0,1]$, we have implicitly characterized all combinations of surplus and revenue that can be obtained through any mechanism. In particular, our results therefore easily yield the mechanisms that maximize surplus subject to a revenue constraint, i.e., dissolution mechanisms that solve

$$\max_{\mathbf{s},\mathbf{t}} \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) s_i(\mathbf{X})] \quad \text{s.t.} \ \sum_{i \in \mathcal{N}} E[t_i(\mathbf{X})] \geq K, \quad \text{(IC), and (IR),} \tag{13}$$

where $K \in \mathbb{R}$. $K = 0$ corresponds to the budget balance requirement commonly studied in the literature. $K > 0$ can be interpreted as the cost of dissolving a partnership, which from the partners' perspective can equivalently be thought of as a deadweight loss resource cost or a transfer to a designer, whereas $K < 0$ describes cases where dissolution (trade) is to some extent subsidized.

Let $(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{t}^{\mathbf{r}}_{\alpha})$ denote the optimal dissolution mechanisms that maximize W_α. Note that because the interim expected shares $S^{\mathbf{r}}_{\alpha,i}(x_i)$ are uniquely pinned down by Theorem 1, all optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{t}^{\mathbf{r}}_{\alpha})$ for a given α generate the same expected surplus $W_0(\alpha, \mathbf{r}) := \sum_{i\in\mathcal{N}} E[s^{\mathbf{r}}_{\alpha,i}(\mathbf{X})v_i(\mathbf{X})]$ and revenue $W_1(\alpha, \mathbf{r}) := \sum_{i\in\mathcal{N}} E[t^{\mathbf{r}}_{\alpha,i}(\mathbf{X})]$.[19]

As $\alpha = 1$ means maximizing revenue, $W_1(1, \mathbf{r}) > 0$ is the highest revenue that can be generated by any dissolution mechanism $(\mathbf{s}, \mathbf{t})$. Hence, the problem (13) does not have a solution if $K > W_1(1, \mathbf{r})$.

From now on, we assume $K \le W_1(1, \mathbf{r})$. Letting $\lambda \ge 0$ denote the Lagrange multiplier on the revenue constraint, we obtain the Lagrangian for problem (13) as

$$\sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})s_i(\mathbf{X})] - \lambda\Big(K - \sum_{i\in\mathcal{N}} E[t_i(\mathbf{X})]\Big) = (1+\lambda)W_{\frac{\lambda}{1+\lambda}}(\mathbf{s}, \mathbf{t}) - \lambda K.$$

It follows that the optimal dissolution mechanisms for an appropriately chosen revenue weight $\alpha = \frac{\lambda}{1+\lambda}$ solve problem (13). More precisely, the appropriate revenue weight is the smallest α that yields at least revenue K, i.e.,

$$\alpha^*(K, \mathbf{r}) := \min\{\alpha \in [0,1] : W_1(\alpha, \mathbf{r}) \ge K\}. \tag{14}$$

If $\alpha^*(K, \mathbf{r}) = 0$, the revenue constraint in (13) is not binding and the first best, i.e., dissolving ex post efficiently, is feasible. Otherwise, the constraint is binding. Then, the second-best mechanisms that maximize surplus subject to generating revenue K are equal to the optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{s}^{\mathbf{r}}_{\alpha})$ of Theorem 1 for $\alpha = \alpha^*(K, \mathbf{r})$.

Note that the literature on partnership dissolution initiated by Cramton, Gibbons, and Klemperer (1987) has almost exclusively focused on the question whether ex post efficient dissolution is possible under budget balance, i.e., whether the first best is feasible for $K = 0$. A major contribution of our analysis is that we provide the second-best dissolution mechanism for all those cases where the first best is not feasible, allowing, in addition, also for non-zero revenue constraints.

3.4 Simple Mechansims and Implementation

Dissolution mechanisms used in practice typically have simpler rules than the optimal direct mechanisms we have characterized. For example, in the Texas shootout (or buy-sell

[19]E.g. using Lemma 1, one can show that $W_0(\mathbf{s}, \mathbf{t}) = \sum_{i\in\mathcal{N}} E\big[(S_i(X_i) - r_i)(X_i - \eta_i(X_i)) + r_i v_i(\mathbf{X})\big]$.

clause), a commonly used procedure for breaking up bilateral business partnerships, one partner proposes a per-unit price and the other one decides whether to sell his share or buy out the other partner at that price (McAfee, 1992; de Frutos and Kittsteiner, 2008; Brooks, Landeo, and Spier, 2010). Under private information, the Texas shootout is typically inefficient, even in settings where ex post efficient dissolution would be feasible. An simple alternative that usually performs better is to run a standard auction among partners, with the highest bidder buying out the others at the price determined in the auction. Under private values, identical distributions, and equal shares, such an auction dissolves efficiently (Cramton, Gibbons, and Klemperer, 1987). Under non-identical distributions or unequal shares, while still satisfying individual rationality, the resulting allocation is inefficient, as shown in Wasser (2013) for the bilateral case. For positively interdependent valuations and equal shares, Kittsteiner (2003) showed that partners suffer from both a winner's and a loser's curse, which may result in a violation of individual rationality. Giving veto rights to agents fixes this problem at the cost of losing efficiency.

The optimal dissolution mechanisms of Theorem 1 are an important benchmark for assessing how severe the inefficiencies are of simple mechanisms such as those just discussed. Moreover, our results provide some guidance for improving the design of simple mechanisms. For the important special case of symmetric bilateral partnerships, we propose in the following a simple mechanism with appealing features for practical use that implements the optimal mechanism.

A simple dissolution procedure Consider a symmetric bilateral partnership, i.e., $n = 2$, $F_1 = F_2$, and $r_1 = r_2 = \frac{1}{2}$. Moreover, suppose $\eta'(x) > -1$ for all x (i.e., interdependence is not strongly negative). This setting allows for a particularly simple and intuitive implementation of the optimal dissolution mechanism as a combination of posted prices and standard auctions. For a given revenue weight α, the optimal allocation rule of Theorem 1 for this setting is based on ironed virtual types with $z_1^* = z_2^* = z^*$ and leaves ownership equally split in case of a tie. To simplify the notation, let $\omega := \omega_{\alpha,i}(z^*)$ denote each agent's critical worst-off type and let $[\underline{\omega}, \overline{\omega}]$ denote the interval of worst-off types, where $\underline{\omega} := (\psi_{\alpha,i}^S)^{-1}(z^*)$ and $\overline{\omega} := (\psi_{\alpha,i}^B)^{-1}(z^*)$.

The simple dissolution procedure works as follows. First, the designer announces four per-unit prices: a standard and a modified sell price, denoted respectively p^S and $\hat{p}^S$, as well as a standard and a modified buy price, denoted p^B and $\hat{p}^B$. These prices are

$$p^S := E[v_i(\omega, X_j)|\overline{\omega} < X_j], \qquad p^B := E[v_i(\omega, X_j)|X_j < \underline{\omega}],$$
$$\hat{p}^S := E[v_i(\underline{\omega}, X_j)|\underline{\omega} \le X_j \le \overline{\omega}], \qquad \hat{p}^B := E[v_i(\overline{\omega}, X_j)|\underline{\omega} \le X_j \le \overline{\omega}].$$

		Agent 2 SELL	Agent 2 HOLD	Agent 2 BUY
Agent 1	SELL	both buy at p^B & reverse auction	1 sells at $\hat{p}^S$, 2 buys at p^B	1 sells at p^S, 2 buys at p^B
Agent 1	HOLD	1 buys at p^B, 2 sells at $\hat{p}^S$	no trade	1 sells at p^S, 2 buys at $\hat{p}^B$
Agent 1	BUY	1 buys at p^B, 2 sells at p^S	1 buys at $\hat{p}^B$, 2 sells at p^S	both sell at p^S & forward auction

Figure 3: A simple dissolution procedure for symmetric bilateral partnerships.

Then, each agent is asked whether he prefers to SELL, HOLD, or BUY. If both agents request something different or if both request HOLD, their requests are called *compatible*. Otherwise, i.e., if both request BUY or both request SELL, their requests are called *incompatible*. If the agents' requests are compatible, shares are traded at fixed prices in the required direction (unless both request HOLD): If agent i requests SELL and agent j BUY, i sells his share at p^S to the designer who then sells it to j at p^B, resulting in revenue of $\frac{1}{2}(p^B - p^S)$ for the designer; if agent i requests HOLD while agent j requests SELL (BUY), then agent i buys j's (sells his) share at the standard price p^B (p^S) whereas agent j's sells (buys) at the modified price $\hat{p}^S$ ($\hat{p}^B$); and if both agents request HOLD, they keep their shares and no payments are made.

Incompatible requests are resolved using a standard auction. If both agents request BUY, the designer first buys the agents' shares at price p^S and then sells the entire object through an open ascending forward auction, with the price starting at $v_i(\overline{\omega}, \overline{\omega})$. If both agents request SELL, the designer first short sells share $\frac{1}{2}$ to each agent at price p^B and then buys back one unit through an open descending reverse auction, starting at price $v_i(\underline{\omega}, \underline{\omega})$. Figure 3 summarizes the procedure.

The following proposition asserts that the simple dissolution procedure implements an optimal dissolution mechanism of Theorem 1.

Proposition 1. *The simple dissolution procedure has a perfect Bayesian equilibrium in which agent $i = 1, 2$ chooses SELL if $x_i < \underline{\omega}$, HOLD if $x_i \in [\underline{\omega}, \overline{\omega}]$, and BUY if $x_i > \overline{\omega}$ and, in case of an auction, agent i drops out when the price reaches $v_i(x_i, x_i)$. Moreover, the resulting allocation and payments correspond to those of an optimal dissolution mechanism $(\mathbf{s^r}, \mathbf{t^r})$.*

Proof. See Appendix B.2. □

Note that the dissolution procedure is simple in that it often only requires coarse communication and uses a standard auction that preserves the privacy of the winner of the auction when more granular information is required. This simplicity suggests that it may be of practical use for designers of dissolution mechanisms such as courts.

4 Optimal Ownership Structures

The optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ that solve (1) described in Theorem 1 depend on $\mathbf{r}$ and the designer's preference parameter α. A natural question is how the designer's value function $W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ varies with $\mathbf{r}$. This is one of the questions we study in this section, proceeding as follows.

We first show that $W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ is concave in $\mathbf{r}$ in general and Schur-concave for agents who are ex ante identical. Concavity means that the designer prefers any partnership whose ownership structure is a convex combination of $\mathbf{r}^1$ and $\mathbf{r}^2$ when he is indifferent between partnerships $\mathbf{r}^1$ and $\mathbf{r}^2$. Moreover, Schur-concavity means that the designer prefers more equal ownership structures to less equal ones. Such preference rankings are, for example, relevant when the designer – for example, divorce lawyers – can choose which partnership he should help dissolve.

The concavity result further allows us to characterize ownership structures that are optimal for the designer insofar as he would, given the option to do so, impose these on the agents before any private information is realized. Although this thought experiment may sound abstract, there are real-world situations where the designer arguably can both choose the ownership structure and the subsequent dissolution mechanism. A case in point is the recently concluded incentive auction. In 2012, US Congress defined broadcasters' property rights over spectrum licenses and authorized the design of the incentive auction, in which revenue generation was an explicit part of the objective (Milgrom, 2017). Likewise, in many court cases, and in particular in countries with weak legal systems and vaguely defined property rights, courts can before or at the beginning of a trial essentially define property rights. Subsequently, they decide how assets should be reallocated, often with the aim or need for extracting rents from the agents.[20]

A related yet subtly different set of questions arise when, at the ex ante stage, the agents rather than the designer can choose the ownership structure, anticipating that at

[20]Similarly, in cap-and-trade pollution trading schemes, Governments typically choose the initial allocation of permits without specific regard to the efficiency of the initial allocation (see, e.g., Schmalensee and Starvins, 2013).

the dissolution stage a certain level of revenue needs to be generated from the dissolution mechanism. We address these questions in Subsection 4.4.

4.1 Designer's Preferences: The Main Result

Having identified the optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ for given initial property rights $\mathbf{r}$ in the preceding section, we are now in a position to study optimal initial ownership structures. In the following we will consider the problem stated in (2), i.e., maximizing $W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ over $\mathbf{r} \in \Delta^{n-1}$.

According to Section 3, we have

$$W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}}) = (1-\alpha) \sum_{i \in \mathcal{N}} E[v_i(\mathbf{X}) r_i] + \max_{\mathbf{s} \in \mathfrak{S}} \min_{\hat{\mathbf{x}} \in [0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}).$$

Since any solution to the max-min problem corresponds to a saddle point $(\mathbf{s}^{\mathbf{r}}, \boldsymbol{\omega}^*)$ of $\widetilde{W}_\alpha$,

$$\begin{aligned} \max_{\mathbf{s} \in \mathfrak{S}} \min_{\hat{\mathbf{x}} \in [0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) &= \min_{\hat{\mathbf{x}} \in [0,1]^n} \max_{\mathbf{s} \in \mathfrak{S}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) \\ &= \min_{\hat{\mathbf{x}} \in [0,1]^n} \bigg\{ - \sum_{i \in \mathcal{N}} r_i E\Big[\psi_{\alpha,i}(X_i, \hat{x}_i)\Big] + \max_{\mathbf{s} \in \mathfrak{S}} E\bigg[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i)\bigg]\bigg\}. \end{aligned}$$

Noting that $E[\psi_{\alpha,i}(X_i, \hat{x}_i)] = (1-\alpha) E[v_i(\mathbf{X})] + \alpha E[v_i(\hat{x}_i, \mathbf{X}_{-i})] + \sum_{j \in \mathcal{N}} E[\eta(X_j)]$, we obtain

$$\begin{aligned} W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}}) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \bigg\{ &- \alpha \sum_{i \in \mathcal{N}} r_i E\Big[v_i(\hat{x}_i, \mathbf{X}_{-i})\Big] \\ &+ \sum_{i \in \mathcal{N}} E[\eta(X_i)] + \max_{\mathbf{s} \in \mathfrak{S}} E\bigg[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i)\bigg]\bigg\}. \end{aligned}$$

In the following, it will be more convenient to represent the standard simplex by $\widehat{\Delta}^{n-1} := \Big\{\mathbf{r} \in [0,1]^{n-1} : \sum_{i=1}^{n-1} r_i \leq 1\Big\}$. Note that using this definition, $(r_1, \ldots, r_{n-1}) \in \widehat{\Delta}^{n-1}$ is equivalent to $\Big(r_1, \ldots, r_{n-1}, 1 - \sum_{i=1}^{n-1} r_i\Big) \in \Delta^{n-1}$. Define the value function $V_\alpha \colon \widehat{\Delta}^{n-1} \to \mathbb{R}$ such that $V_\alpha(\hat{r}_1, \ldots, \hat{r}_{n-1}) = W_\alpha(\mathbf{s}^{\mathbf{r}}, \mathbf{t}^{\mathbf{r}})$ for each $\mathbf{r} = (\hat{r}_1, \ldots, \hat{r}_{n-1}, 1 - \sum_{i=1}^{n-1} \hat{r}_i)$. Hence, for each $\mathbf{r} \in \widehat{\Delta}^{n-1}$,

$$\begin{aligned} V_\alpha(\mathbf{r}) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \bigg\{ &\alpha \sum_{i=1}^{n-1} r_i \Big(E\Big[v_n(\hat{x}_n, \mathbf{X}_{-n})\Big] - E\Big[v_i(\hat{x}_i, \mathbf{X}_{-i})\Big]\Big) - \alpha E\Big[v_n(\hat{x}_n, \mathbf{X}_{-n})\Big] \\ &+ \sum_{i \in \mathcal{N}} E[\eta(X_i)] + \max_{\mathbf{s} \in \mathfrak{S}} E\bigg[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i)\bigg]\bigg\}. \end{aligned}$$

Observe that $V_\alpha(\mathbf{r})$ is the minimum of a family of affine functions of $\mathbf{r}$ (indexed by $\hat{\mathbf{x}}$). Consequently, $V_\alpha(\mathbf{r})$ is concave and differentiable almost everywhere. By the envelope theorem

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} = \alpha\Big(E\big[v_n(\omega_n^*, \mathbf{X}_{-n})\big] - E\big[v_i(\omega_i^*, \mathbf{X}_{-i})\big]\Big) \tag{15}$$

where $\omega_i^* = \omega_{\alpha,i}(z_i^*)$ for $i \in \mathcal{N}$ and $\mathbf{z}^* = \Gamma_n^{-1}\Big(\mathbf{r}, 1 - \sum_{i=1}^{n-1} r_i\Big)$. Note that since each $\omega_{\alpha,i}$ and Γ_n^{-1} are continuous functions, these partial derivatives are continuous. Therefore, V_α is differentiable on $\widehat{\Delta}^{n-1}$.

Theorem 2. *$W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is concave in $\mathbf{r}$. The optimal ownership structures are all $\mathbf{r}^* \in \Delta^{n-1}$ such that $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$ satisfies, for all $i \in \mathcal{N}$ and some Y,*

$$\begin{aligned} E\Big[v_i\Big(\omega_{\alpha,i}(z_i^*), \mathbf{X}_{-i}\Big)\Big] &= Y \quad \textit{if } r_i^* > 0, \\ E\Big[v_i\Big(\omega_{\alpha,i}(\underline{z}), \mathbf{X}_{-i}\Big)\Big] &\geq Y \quad \textit{if } r_i^* = 0. \end{aligned}$$

Proof. That $W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is concave on Δ^{n-1} follows because we have shown that V_α is concave on $\widehat{\Delta}^{n-1}$. Consider the problem of maximizing $V_\alpha(r_1, \ldots, r_{n-1})$ subject to $(r_1, \ldots, r_{n-1}) \in \widehat{\Delta}^{n-1}$. As we maximize a concave and differentiable function over a convex set, a solution exists and can be identified using Kuhn-Tucker conditions. We represent the requirement $(r_1, \ldots, r_{n-1}) \in \widehat{\Delta}^{n-1}$ by the following n inequality constraints: For all $i \in \{1, \ldots, n-1\}$, let λ_i denote the Lagrange multiplier on the constraint $r_i \geq 0$ and let λ_n denote the Lagrange multiplier on the constraint $1 - r_n = \sum_{i=1}^{n-1} r_i \leq 1$. Any solution corresponds to shares and non-negative multipliers satisfying

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} + \lambda_i - \lambda_n = 0 \quad \text{and} \quad \lambda_i r_i = 0 \quad \text{for all } i \in \{1, \ldots, n-1\}$$

as well as $\Big(\sum_{i=1}^{n-1} r_i - 1\Big)\lambda_n = 0$. Using (15) this implies that optimal shares $\mathbf{r}^*$ satisfy

$$\begin{aligned} E\Big[v_i\Big(\omega_{\alpha,i}(z_i^*), \mathbf{X}_{-i}\Big)\Big] &= Y \quad \text{for all } i \in \mathcal{N} \text{ where } r_i > 0, \\ E\Big[v_j\Big(\omega_{\alpha,j}(z_j^*), \mathbf{X}_{-j}\Big)\Big] &\geq Y \quad \text{for all } j \in \mathcal{N} \text{ where } r_j = 0 \end{aligned}$$

where $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$. Finally, note that for all $\mathbf{r}$ and $\mathbf{z} = \Gamma_n(\mathbf{r})$, we have $r_i > (=)\ 0$ if and only if $z_i > (=)\ \underline{z}$, as shown in the proof of Theorem 1 in Appendix A. □

In addition to establishing concavity of the designer's objective, Theorem 2 provides a characterization of the optimal ownership structures using the interim expected valuation of each agent's critical worst-off type induced by the optimal dissolution mechanisms. Every optimal ownership structure equalizes these interim valuations across agents with strictly positive initial shares and induces a higher interim valuation for each agent with an initial share of zero.

Recall that for a given ownership structure $\mathbf{r}$, the interim expected status-quo utility of agent i with type x_i is $r_i E[v_i(x_i, \mathbf{X}_{-i})]$. For all optimal dissolution mechanisms $(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$, each agent i's individual rationality constraint is binding for the interval of worst-off types that results from bunching around the critical worst-off type $\omega_{\alpha,i}(z_i^*)$. Now, consider a marginal transfer of ownership from agent i to agent j. Intuitively, this marginally relaxes i's individual rationality constraint by $E\Big[v_i\Big(\omega_{\alpha,i}(z_i^*), \mathbf{X}_{-i}\Big)\Big]$ and tightens j's individual rationality constraint by $E\Big[v_j\Big(\omega_{\alpha,j}(z_j^*), \mathbf{X}_{-j}\Big)\Big]$. If the former effect were larger than the latter, such a marginal change in $\mathbf{r}$ would improve $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$. Under optimal ownership structures, such marginal changes must be either not beneficial or not feasible (because $r_i = 0$).

In the following two subsections, we explore the implications of Theorem 2. First, we show that if types are identically distributed, concavity of $W_\alpha(\mathbf{s}^\mathbf{r}, \mathbf{t}^\mathbf{r})$ results in symmetric initial shares being always optimal, independent of α. With non-identical distributions, however, optimal ownership structures may be partially or fully concentrated, as we demonstrate further below. Moreover, the identity of the agents at which ownership should be concentrated depends in general on α.

4.2 Schur-concavity and Symmetric Ownership

We will now consider environments where the types of some or all agents are drawn from the same distribution. In this case, the effect of the initial ownership structure on the combination of surplus and revenue that can be achieved through optimal dissolution can be conveniently studied using the theory of majorization.[21] Given two vectors $\mathbf{r}$ and $\mathbf{q}$ with n components we say $\mathbf{r}$ is *majorized* by $\mathbf{q}$, denoted by $\mathbf{r} \prec \mathbf{q}$, if

$$\sum_{i=1}^{k} r_{[i]} \leq \sum_{i=1}^{k} q_{[i]} \quad \text{for } k \in \{1, \dots, n-1\} \quad \text{and} \quad \sum_{i=1}^{n} r_{[i]} = \sum_{i=1}^{n} q_{[i]}$$

where $r_{[1]} \geq \cdots \geq r_{[n]}$ denotes the components of $\mathbf{r} = (r_1, \dots, r_n)$ in decreasing order. Intuitively, $\mathbf{r} \prec \mathbf{q}$ is a notion of the components of $\mathbf{r}$ being more equal (less diverse) than the components of $\mathbf{q}$. A real-valued function ϕ is *Schur-concave* if $\mathbf{r} \prec \mathbf{q}$ implies $\phi(\mathbf{r}) \geq \phi(\mathbf{q})$.

Consider the case where a group $\mathcal{I} \subseteq \mathcal{N}$ of agents have identically distributed types, i.e., $F_i = F_j$ for all $i, j \in \mathcal{I}$. Let $\mathbf{r}_\mathcal{I}$ denote the initial shares of the agents in $\mathcal{I}$ and

[21] For a comprehensive reference, see Marshall, Olkin, and Arnold (2011).

$\mathbf{r}_{-\mathcal{I}}$ those of the remaining agents, such that the ownership structure can be written as $\mathbf{r} = (\mathbf{r}_{\mathcal{I}}, \mathbf{r}_{-\mathcal{I}})$. With the types of the agents in $\mathcal{I}$ being identically distributed, $W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is symmetric in $\mathbf{r}_{\mathcal{I}}$, i.e., $W_\alpha(\mathbf{s}^{(\mathbf{r}_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})}, \mathbf{t}^{(\mathbf{r}_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})}) = W_\alpha(\mathbf{s}^{(\mathbf{r}'_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})}, \mathbf{t}^{(\mathbf{r}'_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})})$ if $\mathbf{r}'_{\mathcal{I}}$ is a permutation of $\mathbf{r}_{\mathcal{I}}$. According to Marshall, Olkin, and Arnold (2011, p. 97) a function is Schur-concave if it is symmetric and concave. Hence, we obtain the following corollary to Theorem 2.

Corollary 2. *Suppose there is a set of agents $\mathcal{I} \subseteq \mathcal{N}$ such that $F_i = F_j$ for all $i, j \in \mathcal{I}$. Then, $\hat{W}_\alpha(\mathbf{r}_{\mathcal{I}}) := W_\alpha(\mathbf{s}^{(\mathbf{r}_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})}, \mathbf{t}^{(\mathbf{r}_{\mathcal{I}},\mathbf{r}_{-\mathcal{I}})})$ is Schur-concave in $\mathbf{r}_{\mathcal{I}}$.*

Schur-concavity in $\mathbf{r}_{\mathcal{I}}$ implies that when holding $\mathbf{r}_{-\mathcal{I}}$ fixed, $W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is maximized at the ownership structure that assigns the same share $r_i = \frac{1}{|\mathcal{I}|}(1 - \sum_{j \notin \mathcal{I}} r_j)$ to each $i \in \mathcal{I}$.

The symmetric treatment of ex ante symmetric agents is consistent with the US Congress's decision in the lead-up to the incentive auction to give identical property rights to all broadcasters, equal to $1/m$ for each broadcaster, where m is the number of licences, and property rights of 0 to all telecom companies.

Note that Corollary 2 also applies when all agents' types are identically distributed, a special case that has received considerable attention in the literature. For this case, the optimal initial ownership structures can be further characterized as follows.

Corollary 3. *Suppose $F_i = F$ for all $i \in \mathcal{N}$. Then, the optimal initial shares are all*

$$\mathbf{r}^* \in \Gamma_n(z^*, \ldots, z^*) = \left\{ \mathbf{r} \in \Delta^{n-1} : \mathbf{r} \prec \mathbf{r}^\alpha \right\}$$

where z^* is the unique solution to

$$\sum_{i \in \mathcal{N}} \left(G_\alpha^S(z^*) \right)^{n-i} \left(G_\alpha^B(z^*) \right)^{i-1} = 1$$

and where $\mathbf{r}^\alpha := (r_1^\alpha, \ldots, r_n^\alpha)$ with $r_i^\alpha := \left(G_\alpha^S(z^*) \right)^{n-i} \left(G_\alpha^B(z^*) \right)^{i-1}$ for all $i \in \mathcal{N}$.

Proof. See Appendix B.3. □

If $F_i = F$ for all $i \in \mathcal{N}$, Schur-concavity implies that $W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is minimized when ownership is concentrated at one agent ($r_i = 1$ for one i) whereas it is maximized for equal ownership ($r_i = \frac{1}{n}$ for all i). Moreover, for all $\mathbf{r} \prec \mathbf{r}^\alpha$ the optimal allocation rule differs from that for initial shares $(\frac{1}{n}, \ldots, \frac{1}{n})$ only with respect to the tie-breaking rule. As the tie-breaking rule does not affect the objective, $W_\alpha(\mathbf{s^r}, \mathbf{t^r})$ is maximized not only by equal initial shares, but by all $\mathbf{r} \prec \mathbf{r}^\alpha$, i.e., by all initial shares in a convex subset of Δ^{n-1}. Increasing α increases the difference between G_α^S and G_α^B. In turn, the components of $\mathbf{r}^\alpha$ become more spread out, which makes the set of optimal initial shares larger.

4.3 Asymmetric Ownership

Having established that equal ownership is optimal in the special case of identically distributed types, we now turn to studying the optimality of concentrated ownership when type distributions vary across agents.

Maximal revenue under ex post efficiency as $\alpha \to 0$ Let us first assume that the revenue weight α is very small. As $\alpha \to 0$, every optimal dissolution mechanism approaches a mechanism with the ex post efficient allocation rule and transfers that maximize revenue under this allocation rule. The optimal ownership structure for $\alpha \to 0$ hence yields the initial shares that allow for the highest revenue under ex post efficient allocation (put differently, these shares minimize the subsidy required for efficient trade). Note that $\omega_{\alpha,i}(z_i^*) \in \Omega_i(\mathbf{s}^\mathbf{r})$ for $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ for all $\mathbf{r}$. As $\alpha \to 0$, $\mathbf{s}^\mathbf{r}$ approaches the ex post efficient allocation rule and $\Omega_i(\mathbf{s}^\mathbf{r})$ shrinks to the singleton $\tilde{\omega}_{0,i}(r_i)$ that solves $\prod_{j \neq i} F_j(\tilde{\omega}_{0,i}) = r_i$, so that $\tilde{\omega}_{0,i}(r_i)$ is agent i's unique worst-off type under the ex post efficient allocation rule. Theorem 2 then yields for the optimal ownership structure under $\alpha = 0$ the initial shares $\mathbf{r}^*$ such that, for all $i \in \mathcal{N}$ and some Y,

$$\tilde{\omega}_{0,i}(r_i^*) + \sum_{j \neq i} E[\eta(X_j)] = Y \quad \text{if } r_i^* > 0, \tag{16}$$

$$\sum_{j \neq i} E[\eta(X_j)] \geq Y \quad \text{if } r_i^* = 0. \tag{17}$$

For $\alpha = 0$, the optimal ownership structure is unique and can be further characterized as follows.

Corollary 4. *Let $\alpha = 0$ and suppose $E[\eta(X_1)] \geq E[\eta(X_2)] \geq \cdots \geq E[\eta(X_n)]$. The optimal ownership structure $\mathbf{r}^*$ is unique: r_1^* is the solution to*

$$r_1^* + \sum_{i=2}^{n} \prod_{j \neq i} F_j\Big(\tilde{\omega}_{0,1}(r_1^*) - E[\eta(X_1)] + E[\eta(X_i)]\Big) = 1$$

and for $i > 1$,

$$r_i^* = \prod_{j \neq i} F_j\Big(\tilde{\omega}_{0,1}(r_1^*) - E[\eta(X_1)] + E[\eta(X_i)]\Big).$$

There is a $k \in \mathcal{N}$ such that $r_i^* > 0$ for $i \leq k$ and $r_i^* = 0$ for $i > k$.

Proof. See Appendix B.4. □

For private values (where $\eta'(x) = 0$ for all x), the revenue maximizing ownership structure under ex post efficiency has been obtained by Che (2006) and Figueroa and

Skreta (2012). Corollary 4 generalizes the results of these authors to interdependent values (where $\eta'(x) \neq 0$). In contrast to the private values case where, as observed by Figueroa and Skreta (2012), all agents have strictly positive shares, the asymmetry in $E[\eta(X_i)]$ under interdependent values may result in an extreme ownership structure where some of the agents own zero shares.

Proposition 2. *Let $\alpha = 0$ and consider two agents $i, j \in \mathcal{N}$.*

(i) If $E[\eta(X_i)] \geq E[\eta(X_j)]$ and $F_i(x) < F_j(x)$ for all $x \in (0,1)$, then either $1 > r_i^* > r_j^* \geq 0$ or $r_i^* = r_j^* = 0$.

(ii) If $E[\eta(X_i)] \leq E[\eta(X_j)] - 1$, then $0 = r_i^* \leq r_j^* \leq 1$.

Proof. See Appendix B.5. □

Suppose F_i first-order stochastically dominates F_j. Under private values, we know from Che (2006) and Figueroa and Skreta (2012) that it is optimal to assign a higher initial share to agent i than to agent j. Part (i) of Proposition 2 shows that this generalizes to positively interdependent types where $\eta'(x) \geq 0$ for all x (which is sufficient for $E[\eta(X_i)] \geq E[\eta(X_j)]$). However, if the interdependence is sufficiently negative such that Part (ii) of Proposition 2 applies (which requires $\eta'(x) < -1$ for at least some x), agent i optimally receives an initial share of zero whereas agent j may receive a nonzero share.

$F_i < F_j$ implies that i's worst-off type $\hat{\omega}_{0,i}(r)$ is strictly lower than j's worst-off type $\hat{\omega}_{0,j}(r)$ for equal initial shares. Moreover, under positive interdependence, for the same type x, i's interim valuation $E[v_i(x, \mathbf{X}_{-i})]$ is less than j's interim valuation $E[v_j(x, \mathbf{X}_{-j})]$. Hence, to equalize interim valuations of worst-off types, a larger share has to be assigned to agent i. In contrast, under negative interdependence $E[v_i(x, \mathbf{X}_{-i})] > E[v_j(x, \mathbf{X}_{-j})]$, which may outweigh the asymmetry in worst-off types and induce $r_i^* < r_j^*$.

Note that Proposition 2 continues to hold for sufficiently small but positive values of α, as in this case the optimal allocation rule is still close to the ex post efficient one.

Extreme ownership structures Let us return to the general case where $\alpha \in [0,1]$. The following proposition informs us on the (sub)optimality of extreme ownership structures, i.e., of property rights that are fully concentrated at one agent i.

Proposition 3. *(i) If $r_i^* = 1$ for $\alpha = \hat{\alpha}$, then $r_i^* = 1$ for all $\alpha > \hat{\alpha}$. If $r_i^* < 1$ for $\alpha = \hat{\alpha}$, then $r_i^* < 1$ for all $\alpha < \hat{\alpha}$.*

(ii) $r_i^* = 1$ only if $E[v_i(\mathbf{X})] < E[v_j(\mathbf{X})]$ for all $j \neq i$. $r_i^* = 1$ if $E[\eta(X_i)] - E[\eta(X_j)] \geq 1$ for all $j \neq i$.

(iii) If $\eta'(x) \in (-1, 1)$ for all x, then $r_i^* < 1$ for all $i \in \mathcal{N}$ and $\alpha \in [0, \tilde{\alpha})$ for some $\tilde{\alpha} > 0$.

Proof. See Appendix B.6. □

According to (i) in Proposition 3, a higher revenue weight α makes the optimality of an extreme ownership structure more likely. Independent of α, (ii) shows that if an extreme ownership structure is optimal, then full ownership is assigned to the agent with the lowest ex ante expected valuation for the object. Moreover, (ii) provides a sufficient condition for the optimality of an extreme ownership structure (requiring sufficiently asymmetric type distributions and negative interdependence). As demonstrated by (iii), unless the interdependence of valuations is strongly negative, optimal ownership structures are always non-extreme for α small enough.

Bilateral partnerships We now turn to studying the optimal ownership structures for bilateral partnerships. The tractability of the bilateral case allows for a more detailed characterization of the optimal initial shares identified in Theorem 2. In particular, we show as part of the following proposition that all optimal ownership structures in a given environment correspond to a unique vector of ironing parameters $\mathbf{z}$ for the associated dissolution mechanisms.

Proposition 4. *Suppose $n = 2$.*

(i) If $\omega_{\alpha,1}(\underline{z}) - E[\eta(X_1)] \geq \omega_{\alpha,2}(\overline{z}) - E[\eta(X_2)]$, then the extreme ownership structure $(r_1^*, r_2^*) = (0, 1)$ is optimal.

(ii) If $\omega_{\alpha,1}(\overline{z}) - E[\eta(X_1)] \leq \omega_{\alpha,2}(\underline{z}) - E[\eta(X_2)]$, then the extreme ownership structure $(r_1^*, r_2^*) = (1, 0)$ is optimal.

(iii) Otherwise, all ownership structures $(r_1^*, r_2^*) \in \Gamma_2(\mathbf{z}^*) \cap \Delta^1$ are optimal and non-extreme, where $\mathbf{z}^*$ is the unique $\mathbf{z}^* \in (\underline{z}, \overline{z})^2$ that satisfies

$$\omega_{\alpha,1}(z_1^*) - E[\eta(X_1)] = \omega_{\alpha,2}(z_2^*) - E[\eta(X_2)] \quad \textit{and} \quad \Gamma_2(\mathbf{z}^*) \cap \Delta^1 \neq \varnothing.$$

Proof. See Appendix B.7. □

(i) and (ii) in Proposition 4 describe corner solutions in which one agent obtains full ownership while (iii) captures the situations in which optimal initial shares are interior.

In the remainder of this subsection, we will focus on a class of examples of bilateral partnerships. Among other things, these examples illustrate that under optimal

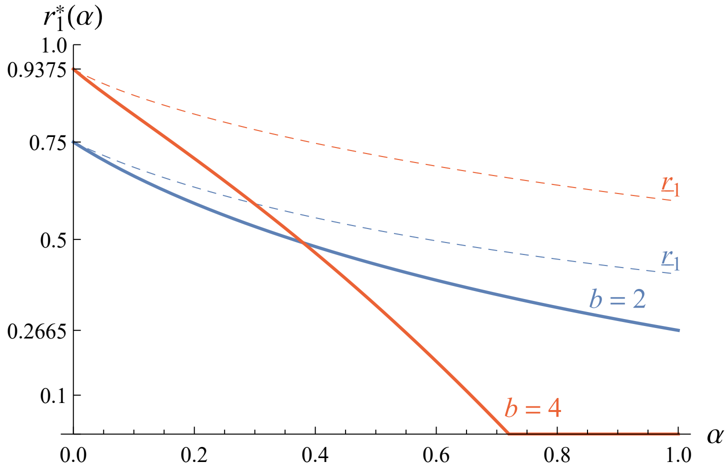


Figure 4: Optimal bilateral ownership structure $r_1^*(\alpha)$ under private values ($e = 0$) for $b = 2$ in blue and $b = 4$ in red.

ownership, the identity of the majority owner may change with α and that if revenue is important, extreme ownership structures may be obtained independent of whether values are positively interdependent, private, or negatively interdependent.

A class of examples Consider bilateral partnerships with the following two-parameter family of specifications for type distributions and value interdependence:

$$F_1(x) = x^b, \quad F_2(x) = 1 - (1 - x)^b, \quad \text{and} \quad \eta(x) = ex, \quad \text{where } b > 1 \text{ and } e < 1. \quad (18)$$

Under these assumptions, $E[v_1(\mathbf{X})] = \frac{b+e}{1+b} > \frac{1+eb}{1+b} = E[v_2(\mathbf{X})]$ and $F_1(x) < F_2(x)$ for all x. Agent 1 is hence the stronger agent in the sense that F_1 first-order stochastically dominates F_2, and the higher is b, the more pronounced is this dominance. Note that according to (ii) of Proposition 3, it is never optimal to assign full ownership to agent 1.

Figure 4 shows the optimal share $r_1^* = r_1^*(\alpha)$ of agent 1 as a function of α with private values (i.e., $e = 0$) for the case with $b = 2$ depicted in blue and for the case with $b = 4$ depicted in red. Figure 4 also depicts how $\underline{r}_1$ changes with α. Note that $r_1^*(\alpha)$ is decreasing, unique, and always below $\underline{r}_1$, which means that under optimal ownership, the optimal dissolution mechanism features ironing parameters $z_1^* < z_2^*$ (cf. Subsection 3.2).

When α is small, the optimal ownership structure favors the strong agent. However, as the weight on revenue increases, the strong agent is eventually discriminated against and ultimately obtains a smaller share than the weak agent for large α. In the second case, this goes so far that the optimal ownership structure gives the strong agent an ownership share of 0 for α in excess of three quarters. Hence, extreme ownership struc-

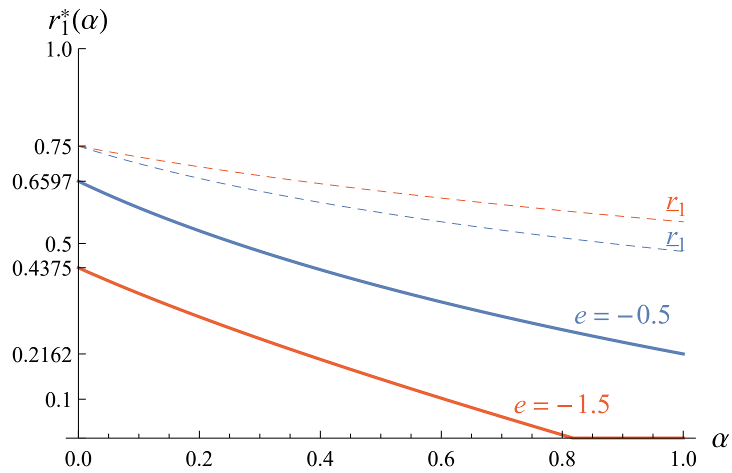


Figure 5: Optimal bilateral ownership structure $r_1^*(\alpha)$ with $b = 2$ and negative interdependence for $e = -0.5$ in blue and $e = -1.5$ in red.

tures, which are at the heart of the bilateral trade model of Myerson and Satterthwaite (1983), can be optimal even with private values, provided α is large enough and agents are ex ante sufficiently different. Intuitively, giving more or even full initial ownership to the agent who is expected to have the lower valuation increases the potential gains from trade at the dissolution stage. Initially favoring the weak agent is optimal if generating revenue, which is extracted from the gains from trade, is important.

Figure 5 illustrates the optimal ownership structure for the distributions with $b = 2$ and two levels of negative interdependence: $e = -0.5$ in blue and $e = -1.5$ in red. For a given α, the stronger is the negative interdependence, the smaller is the strong agent's optimal share. As in the private values case, $r_1^*(\alpha)$ is decreasing and always below $\underline{r}_1$. Note that for even stronger negative interdependence where $e \leq \frac{b+1}{b-1} = -3$, the sufficient condition in (ii) of Proposition 3 is satisfied, resulting in $r_1^*(\alpha) = 0$ for all α.

Finally, we consider positively interdependent values in Figure 6, assuming $e = 0.35$. The optimal ownership structures $r_1^*(\alpha)$ are displayed, together with $\underline{r}_1$ and $\overline{r}_1$ again for $b = 2$ in blue. In contrast to above, $r_1^*(\alpha)$ is now increasing, unique, and above $\overline{r}_1$ (implying $z_1^* > z_2^*$) for small α. Moreover, there is a unique α for which all $r_1^* \in r_1^*(\alpha) = [\underline{r}_1, \overline{r}_1]$ are optimal ownership structures.[22] For higher values of α, $r_1^*(\alpha)$ has similar properties as in the other figures: it is decreasing, unique, and below $\underline{r}_1$. Again, it is optimal to make the strong agent the majority owner for small α and the minority

[22] For bilateral environments where $F_2(x) = 1 - F_1(1 - x)$ for all x and $\eta(x) = ex$ for some $e < 1$, as is satisfied here, the virtual type distributions satisfy $G^B_{\alpha,2}(z) = 1 - G^S_{\alpha,1}(1 - e - z)$ and $G^S_{\alpha,2}(z) = 1 - G^B_{\alpha,1}(1 - e - z)$ for all z. For the optimal dissolution mechanism of Corollary 1, this symmetry property implies that $z_1^* = z_2^* = \frac{1-e}{2}$ for all $r_1 \in [\underline{r}_1, \overline{r}_1]$. Hence, if the optimal ownership structures are such that $z_1^* = z_2^*$, then the entire interval $[\underline{r}_1, \overline{r}_1]$ is optimal.

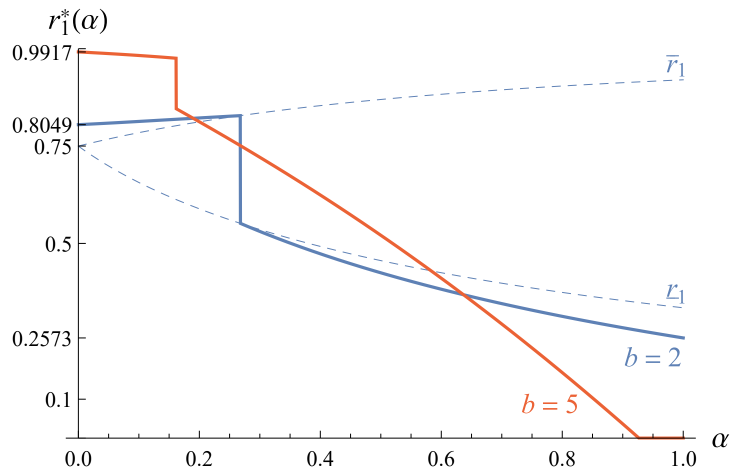


Figure 6: Optimal bilateral ownership structure $r_1^*(\alpha)$ under positive interdependence with $e = 0.35$ for $b = 2$ in blue and $b = 5$ in red.

owner for large α. Displayed in red, Figure 6 also shows the optimal ownership structures $r_1^*(\alpha)$ for $b = 5$. As in the other figures, when the asymmetry in type distributions is sufficiently pronounced, it becomes optimal to assign full ownership to the weaker agent if generating revenue is important.

4.4 Ownership Structures Chosen by Agents

Another natural question is what initial ownership structure the agents would choose ex ante, that is, before any private information is realized, anticipating the costs associated with subsequent reallocation of property rights due to incentive compatibility and individual rationality constraints and, possibly, additional costs (such as court costs). Because there is no private information at this ex ante stage, it is natural to assume that the partners choose an ownership structure that maximizes their joint expected social surplus.[23] We now briefly analyze this problem, assuming that, to reallocate ownership is only possible at fixed and known cost. As mentioned, this could equivalently be the fee that a court or matchbreaker will charge.[24]

Suppose there are two stages. In the first stage, which takes place ex ante (before agents learn their signals $\mathbf{x}$), agents bargain efficiently over the allocation of initial prop-

[23]Without modeling this bargaining game, one would expect in light of the Coase Theorem that the agents could transfer utility in order to reach an agreement.

[24]We assume, however, that K does not depend on the ownership structure $\mathbf{r}$. That is, we assume that the designer of the dissolution mechanism cannot price-discriminate across partnerships, for example, because he has to announce his fee before he observes the ownership structure. While interesting, we leave for future research the analysis of optimal ownership structures chosen by agents who anticipate that a price-discriminating designer wants to maximize revenue.

erty rights $\mathbf{r}$. In the second stage, each agent privately learns his signal x_i. Now agents may either reallocate property rights or stick with their initial shares. Reallocation costs $K \in \mathbb{R}$, i.e., the agents must use a mechanism that generates at least revenue K.

Note that the second stage corresponds to the situation discussed in Subsection 3.3: If feasible, property rights are reallocated using a mechanism that maximizes surplus subject to raising revenue K. Recall that we have defined $W_0(\alpha, \mathbf{r}) = \sum_{i \in \mathcal{N}} E[s^{\mathbf{r}}_{\alpha,i}(\mathbf{X}) v_i(\mathbf{X})]$ to be the surplus and $W_1(\alpha, \mathbf{r}) = \sum_{i \in \mathcal{N}} E[t^{\mathbf{r}}_{\alpha,i}(\mathbf{X})]$ to be the revenue generated by all optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{t}^{\mathbf{r}}_{\alpha})$ of Theorem 1 that maximize W_α under initial ownership structure $\mathbf{r}$.

If $W_1(1, \mathbf{r}) < K$, reallocating property rights at cost K is not feasible, resulting in expected surplus $\sum_{i \in \mathcal{N}} r_i E[v_i(\mathbf{X})]$. Otherwise, second-best dissolution mechanisms are equal to optimal mechanisms for $\alpha = \alpha^*(K, \mathbf{r})$, as defined in (14). In this case, the expected surplus is $W_0(\alpha^*(K, \mathbf{r}), \mathbf{r})$. Note that since the reallocation mechanism satisfies (IR), all agents prefer reallocation whenever it is feasible.

In the first stage, agents anticipate the effect of the initial shares on the second stage. The sum of the agents' payoffs from agreeing on $\mathbf{r}$ initially is

$$P(K, \mathbf{r}) := \begin{cases} W_0(\alpha^*(K, \mathbf{r}), \mathbf{r}) - K & \text{if } W_1(1, \mathbf{r}) \geq K, \\ \sum_{i \in \mathcal{N}} r_i E[v_i(\mathbf{X})] & \text{if } W_1(1, \mathbf{r}) < K, \end{cases}$$

i.e., the value of the final allocation minus the costs K in case of reallocation. Bargaining efficiently in the absence of private information, the agents hence will agree on an ownership structure that solves $\max_{\mathbf{r} \in \Delta^{n-1}} P(K, \mathbf{r})$. Let

$$R(K) := \arg\max_{\mathbf{r} \in \Delta^{n-1}} P(K, \mathbf{r})$$

denote the set of all agent-optimal ownership structures given reallocation cost K.

In the following, we characterize $R(K)$ using our results on optimal ownership structures for fixed α developed in the preceding subsections. To this end, let $R^*(\alpha)$ denote the set of optimal ownership structures characterized in Theorem 2 for a given $\alpha > 0$, i.e., $R^*(\alpha) := \arg\max_{\mathbf{r} \in \Delta^{n-1}} W_\alpha(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{t}^{\mathbf{r}}_{\alpha})$.

First, suppose $K > \overline{K} := W_1(1, \mathbf{r}^1)$ for $\mathbf{r}^1 \in R^*(1)$. In this case, reallocation costs are prohibitively large: even if agents were to choose revenue-maximizing initial shares, reallocation would still not be feasible. Hence, agents choose an ownership structure that maximizes the expected surplus in absence of reallocation, i.e., some

$$\hat{\mathbf{r}} \in \hat{R} := \arg\max_{\mathbf{r}\in\Delta^{n-1}} \sum_{i\in\mathcal{N}} r_i E[v_i(\mathbf{X})],$$

resulting in surplus (and joint payoff) $\hat{P} := \max_{i\in\mathcal{N}} E[v_i(\mathbf{X})]$.

If $K \leq \overline{K}$, reallocation of shares is feasible provided that the agents choose the initial property rights appropriately. However, doing so is in the agents' interest if and only if

$$\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}),\mathbf{r}) - K \geq \hat{P}, \tag{19}$$

i.e., if and only if the best ownership structure that renders reallocation feasible yields a higher payoff than the best fixed ownership structure. Note that the LHS of (19) is strictly decreasing in K since $W_0(\alpha^*(K',\mathbf{r}),\mathbf{r}) \geq W_0(\alpha^*(K'',\mathbf{r}),\mathbf{r})$ for all $K' < K''$ and $\mathbf{r}$. Consequently, (19) is satisfied for all $K \leq \hat{K}$, where

$$\hat{K} := \max\left\{K \leq \overline{K} \colon \max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}),\mathbf{r}) - K \geq \hat{P}\right\}.$$

For $K > \hat{K}$, however, agents choose some $\hat{\mathbf{r}} \in \hat{R}$ and abstain from reallocating.[25]

In the following, we assume $K \leq \hat{K}$. Let $\mathbf{r}^0$ denote the revenue-maximizing ownership structure under ex post efficiency as characterized by Corollary 4. Moreover, define $\underline{K} := W_1(0,\mathbf{r}^0)$. If $K \leq \underline{K}$, generating revenue K and reallocating ex post efficiently is feasible under appropriately chosen initial ownership. As doing so results in the highest possible surplus, any ownership structure $\mathbf{r}$ such that $\alpha^*(K,\mathbf{r}) > 0$ must be strictly suboptimal for the agents. Hence, if $K \leq \underline{K}$, agents optimally choose an initial ownership structure in $\{\mathbf{r} \in \Delta^{n-1} : W_1(0,\mathbf{r}) \geq K\}$ and then reallocate ex post efficiently.

If $K > \underline{K}$, ex post efficient reallocation is not feasible under any initial ownership. In this case, all $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}),\mathbf{r})$ will induce reallocation with an optimal dissolution mechanism for $\alpha^*(K,\mathbf{r}_K) > 0$. The following lemma shows that the ownership structures that are optimal for the agents when they plan to reallocate are all contained in the set of optimal ownership structures $R^*(\alpha)$ for one specific α.

Lemma 2. *Suppose $K \in (\underline{K}, \overline{K}]$. Then, $\arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}),\mathbf{r}) \subseteq R^*(\alpha(K))$, where $\alpha(K)$ is the unique $\alpha \in (0,1]$ such that $W_1(\alpha,\mathbf{r}) = K$ for some $\mathbf{r} \in R^*(\alpha)$.*

[25] Note that this does not require that agents can commit to abstain from reallocating since reallocation is in this case not feasible under any $\hat{\mathbf{r}} \in \hat{R}$: On the one hand,

$$\hat{P} > \max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}),\mathbf{r}) - K \geq \max_{\mathbf{r}\in\Delta^{n-1}} W_0(1,\mathbf{r}) - K \geq W_0(1,\hat{\mathbf{r}}) - K.$$

On the other hand, individual rationality ensures that $W_0(1,\hat{\mathbf{r}}) - W_1(1,\hat{\mathbf{r}}) \geq \hat{P}$. Hence, $W_1(1,\hat{\mathbf{r}}) < K$.

Proof. See Appendix B.8. □

According to Lemma 2, if $K \in (\underline{K}, \hat{K}]$, then $R(K) \subseteq R^*(\alpha(K))$, where $\alpha(K)$ is the unique revenue weight that allows for corresponding optimal ownership structures under which the optimal dissolution mechanism generates exactly revenue K. $R(K)$ then consist of those $\mathbf{r} \in R^*(\alpha(K))$ that yield the highest surplus while generating revenue K. Note that for $n = 2$ and for $n \geq 2$ with identically distributed types, Proposition 4 and Corollary 3, respectively, assert that the corresponding $\mathbf{z}^*$ and hence also surplus as well as revenue are the same for all $\mathbf{r} \in R^*(\alpha)$, implying $R(K) = R^*(\alpha(K))$.

The following proposition summarizes our findings.

Proposition 5. *The agent-optimal ownership structures $R(K)$ are as follows:*

(i) If $K \leq \min\{\underline{K}, \hat{K}\}$, $R(K) = \{\mathbf{r} \in \Delta^{n-1} : W_1(0, \mathbf{r}) \geq K\}$ and agents reallocate using an ex post efficient dissolution mechanism.

(ii) If $K \in (\min\{\underline{K}, \hat{K}\}, \hat{K}]$,

$$R(K) = \underset{\mathbf{r} \in R^*(\alpha(K))}{\arg\max}\ W_0(\alpha(K), \mathbf{r}) \ \text{s.t.}\ W_1(\alpha(K), \mathbf{r}) = K$$

and agents reallocate using an optimal dissolution mechanism for $\alpha(K)$.

(iii) If $K \in (\hat{K}, \overline{K}]$, $R(K) = \hat{R}$ and there is no reallocation, although reallocation would be feasible for some $\mathbf{r} \notin \hat{R}$.

(iv) If $K > \overline{K}$, $R(K) = \hat{R}$ and reallocation is not feasible under any $\mathbf{r} \in \Delta^{n-1}$.

Identically distributed types Suppose $F_i = F$ for all $i \in \mathcal{N}$. In this case, the expected surplus from an ex ante fixed ownership structure is independent of the choice of $\mathbf{r}$. Because optimal dissolution mechanisms are individually rational, $W_0(1, \mathbf{r}) - W_1(1, \mathbf{r}) \geq \sum_{i \in \mathcal{N}} r_i E[v_i(\mathbf{X})] = \hat{P}$ for all $\mathbf{r}$. Hence (19) is satisfied even for $K = \overline{K}$, implying $\hat{K} = \overline{K}$. Recall from Corollary 3 that equal shares is an optimal ownership structure for all α and also maximizes revenue under ex post efficiency. Hence, for all K, it is optimal for agents to choose symmetric initial shares and reallocate whenever feasible.

While equal ownership is always contained in $R(K)$, $R(K)$ is not a singleton unless $K = \underline{K}$. For $K < \underline{K}$, the set $R(K)$ continuously decreases in K because a tighter revenue constraint means that there are less initial shares that allow for ex post efficient reallocation. In contrast, for $K \in (\underline{K}, \overline{K})$, the set $R(K)$ continuously increases in K because $\alpha(K)$ is increasing, resulting in more shares being optimal according to Corollary 3. At $K = \overline{K}$, there is a discontinuity since $R(K) = \hat{R} = \Delta^{n-1}$ for all $K > \overline{K}$.

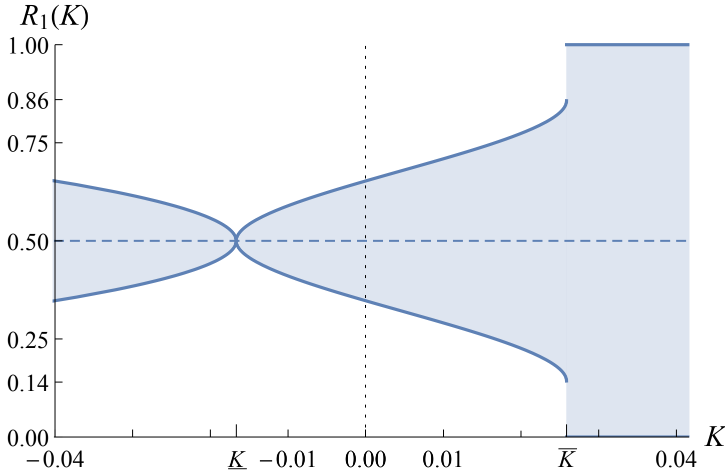


Figure 7: Set $R_1(K)$ of agent-optimal r_1 as a function of K for $n = 2$, $\eta(x) = 0.6x$, and with identically and uniformly distributed types. We have $\underline{K} = -\frac{1}{60} = -0.01667$, $\hat{K} = \overline{K} = \frac{19}{735} = 0.02585$.

Figure 7 illustrates how $R(K)$ changes with K for a bilateral partnership where the types of both agents are drawn from the uniform distribution and $\eta(x) = 0.6$. Displayed is $R_1(K)$, the set of agent-optimal shares for agent 1 (such that $R(K) = \{(r_1, r_2) \in \Delta^1 : r_1 \in R_1(K)\}$). Note that under this specification with positive interdependence of valuations, $\underline{K} < 0$, i.e., ex post efficient dissolution is impossible under any initial ownership structure even if $K = 0$.

Under identically distributed types, equal initial ownership is robust in the sense that it is always agent-optimal independent of K. Equal shares are hence also an optimal choice should there be some uncertainty at the ex ante stage regarding the size of K. How much flexibility agents have in choosing the initial ownership, i.e., the size of $R(K)$, depends non-monotonically on K.

Asymmetric type distributions To study the effects of non-identical type distributions, we turn to a specific parametrization. In the following, we consider a bilateral partnership with $\eta(x) = 0$ (private values) and assume that type distributions are $F_1(x) = x^b$ and $F_2(x) = 1 - (1 - x)^b$ for some $b > 1$ (as in (18) in the preceding subsection). Numerical simulations yield

$$\hat{K} \begin{cases} = \overline{K} & \text{if } b < 1.116, \\ \in (\underline{K}, \overline{K}) & \text{if } 1.116 < b < 1.579, \\ \leq \underline{K} & \text{if } b > 1.579. \end{cases}$$

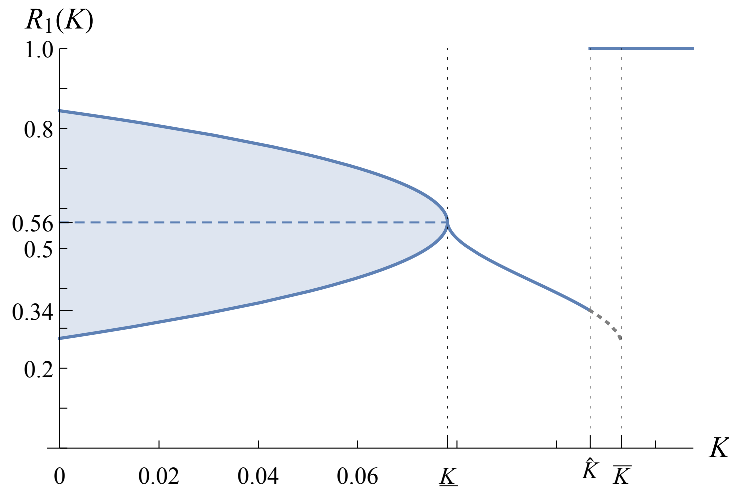


Figure 8: Agent-optimal shares for agent 1 $R_1(K)$ for $b = 1.2$, where $\underline{K} = 0.0781$, $\hat{K} = 0.1068$, and $\overline{K} = 0.1131$.

For b sufficiently close to 1, $\hat{K} = \overline{K}$ as in the case of identically distributed types, i.e., case (iii) of Proposition 5 never applies and agents choose initial shares that induce reallocation whenever feasible. In contrast, for sufficiently large b, $\hat{K} \leq \underline{K}$ such that case of (ii) Proposition 5 never applies: there is either ex post efficient reallocation or no reallocation at all. In particular, fixed ownership is in this case also agent-optimal for the entire range of K where reallocation would be feasible using an optimal dissolution mechanism for some $\alpha > 0$. For intermediate values of b, where $\hat{K} \in (\underline{K}, \overline{K})$, each of the four cases of Proposition 5 applies for some K.

Assume now that $b = 1.2$. Under this specification, $\underline{K} = 0.0781$, $\hat{K} = 0.1068$, and $\overline{K} = 0.1131$. Figure 8 displays the set of agent-optimal initial shares $R_1(K)$. Similar to the case of identically distributed types, for $K < \underline{K}$, $R_1(K)$ is not a singleton and is decreasing in K. The revenue maximizing share under ex post efficiency, which is characterized in Corollary 4 and always contained in $R_1(K)$, is equal to 0.56, favoring the stronger agent. For all $K > \underline{K}$, however, $R_1(K)$ is a singleton: it is first decreasing in K till it reaches 0.34 at $\hat{K}$, favoring the weaker agent, and then jumps to the optimal fixed share of 1 (sole ownership by agent 1), even though for $K \in (\hat{K}, \overline{K}]$ reallocation would be feasible (with the dashed gray line indicated the optimal share for that purpose).

The parameter K can be interpreted as measuring costly frictions for reallocating property rights due to legal uncertainty, lengthy and costly legal procedures, or simply the level of corruption. Our findings suggests such frictions have a profound impact on the structure of businesses, leading to indirect, additional costs. On the one hand, moderate to relatively high frictions may limit the partners' ability to restructure their business in response to changing environments (case (ii) of Proposition 5) or may even prevent them entirely from doing so (cases (iii) and (iv) of Proposition 5). Unless partners are ex ante symmetric, a small increase in frictions K from below to above $\hat{K}$ may drastically change the initially chosen ownership structure: in the example displayed in Figure 8, the stronger partner's position changes from owning a minority share to sole ownership. On the other hand, even for relatively low frictions (case (i) of Proposition 5) where efficient restructuring is expected, an additional indirect costs stems from the partners' limited flexibility in setting up initial ownership (also regarding concerns outside the model), as the set of agent-optimal ownership structures shrinks with K.

5 Conclusions

We fully characterize the optimal dissolution mechanisms for a general partnership model. Beyond identifying optimal break-up procedures for business partnerships, our results broadly provide guidance for the design of trading platforms for homogeneous goods with arbitrary initial endowments. To curb information rents, the optimal mechanisms allocate based on uniquely determined ironed virtual type functions, which for each agent are constant for a (typically interior) range of types while corresponding to virtual costs and valuations, respectively, for lower and higher types. Ceteris paribus, the allocation is biased towards agents with larger initial shares through an upward shift of the range where the virtual type is constant or through favorable treatment in case of a tie. Ex ante heterogeneity in type distributions directly translates to asymmetric virtual types, such that, for example, stronger agents in terms of (reverse) hazard rate dominance are discriminated against.

We also study the effects of changes in initial shares on the outcome of optimal dissolution and characterize optimal initial ownership structures. Symmetric ownership is always optimal under identically distributed types but typically not otherwise. Who should optimally own larger initial shares then depends on the importance of revenue generation, and even extreme ownership structures can be optimal.

Embedding the static problem studied here in a dynamic setup seems a particularly valuable avenue for future research. Another important issue that relates to the allocation of property rights concerns their effects on the incentives to invest (Schmitz, 2002; Segal and Whinston, 2011, 2013), which we have abstracted away from in the present paper. Given the prominent role transaction costs and information play in the theory of the firm, a natural step for further research is to use our methods and results to develop models that permit a unified approach of bargaining within and across firms.

Appendix A: Proof of Theorem 1

We first prove the second part of the theorem, i.e., the statements after the first line, taking as given that the statement in the first line is true. Then we prove the statement in the first line.

Suppose there exists a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ such that $\mathbf{r} \in \Gamma_n(\mathbf{z})$, as stated in the first line of the theorem. It follows that for this unique $\mathbf{z}$ and some tie-breaking rule $\mathbf{a}$, the ironed virtual type allocation rule $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ and the critical types $\Big(\omega_{\alpha,1}(z_1), \ldots, \omega_{\alpha,n}(z_n)\Big)$ constitute a saddle point satisfying (9) and (10), making $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ an optimal allocation rule consistent with (8).

Note that through restricting the definition of Γ_n and the statement of Theorem 1 to $z_i \in [\underline{z}, \overline{z}] = [\psi^S_{\alpha,i}(0), \psi^B_{\alpha,i}(1)] \subset \Big[\psi^B_{\alpha,i}(0), \psi^S_{\alpha,i}(1)\Big]$, we have confined attention to critical types $\omega^*_i \in [\omega_{\alpha,i}(\underline{z}), \omega_{\alpha,i}(\overline{z})] \subset [0, 1]$. This restriction is without loss when looking for optimal allocation rules. As will become apparent below, for $\mathbf{z} = \Gamma_n^{-1}(\mathbf{r})$ we have $z_i = \underline{z}$ if and only if $r_i = 0$. Hence for all $\mathbf{r}$, $z_j > \underline{z}$ for at least one j. Accordingly, $S^{\mathbf{z},\mathbf{a}}_i(\omega_{\alpha,i}(z_i)) = 0$ for all $z_i \leq \underline{z}$. If there is a saddle point involving critical type $\omega^*_i = \omega_{\alpha,i}(\underline{z})$ then there is also a saddle point for each $\omega^*_i \in [0, \omega_{\alpha,i}(\underline{z}))$. However, all these saddle points are equivalent in terms of the implied allocation rule $\mathbf{s}^*$ and i's worst-off types $\Omega_i(\mathbf{s}^*) = \{x_i : S^*_i(x_i) = 0\} = \Big[0, (\psi^B_{\alpha,i})^{-1}(\underline{z})\Big]$. A similar line of argument can be invoked for $z_i \geq \overline{z}$, which only occurs if $r_i = 1$.

From the preceding paragraph, we conclude that whereas there can be multiple saddle points satisfying (9) and (10), the corresponding allocation rule $\mathbf{s}^*$ is unique up to the tie-breaking rule and can be defined as allocating to the greatest ironed virtual type for ironing parameters $\mathbf{z} = \Gamma_n^{-1}(\mathbf{r})$. Whereas the exact specification of the tie-breaking rule may differ, all optimal allocation rules result in the same interim expected shares, which in turn pin down interim expected payments, as explained in the main text.

It remains to prove the first line of the theorem. For any $A \subseteq [\underline{z}, \overline{z}]^n$, let $\Gamma_n(A) = \Big\{\mathbf{y} \in [0, 1]^n : \mathbf{y} \in \Gamma_n(\mathbf{z}) \text{ for some } \mathbf{z} \in A\Big\}$ denote the image of A under Γ_n. To prove that for each $\mathbf{r} \in \Delta^{n-1}$, there is a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ such that $\mathbf{r} \in \Gamma_n(\mathbf{z})$, we will show that Γ_n has the following two properties:

Property 1: For every $\mathbf{y} \in \Gamma_n\Big([\underline{z}, \overline{z}]^n\Big)$, there is a unique $\mathbf{z}$ such that $\mathbf{y} \in \Gamma_n(\mathbf{z})$.

Property 2: $\Delta^{n-1} \subset \Gamma_n\Big([\underline{z}, \overline{z}]^n\Big)$.

Property 1 implies the uniqueness part. It says that every point in the image of Γ_n corresponds to exactly one $\mathbf{z}$. Put differently, the inverse correspondence $\Gamma_n^{-1}(\mathbf{y}) := \Big\{\mathbf{z} \in [\underline{z}, \overline{z}]^n : \mathbf{y} \in \Gamma_n(\mathbf{z})\Big\}$ is singleton valued for all $\mathbf{y} \in \Gamma_n([\underline{z}, \overline{z}]^n)$. *Property 2* implies the existence part. It says that the image of Γ_n contains the standard simplex Δ^{n-1}.

The proof proceeds as follows. After some definitions and preliminary results in Subsection A.1, we show in Subsection A.2 that *Property 1* and *Property 2* hold for $n = 2$. In Subsection A.3, we first uncover the recursive structure of Γ_n. This then allows us to prove by induction that the two properties hold for all n, using $n = 2$ as the base case.

A.1 Preliminaries

Recall the virtual cost distributions $G^S_{\alpha,i}$ and virtual valuation distributions $G^B_{\alpha,i}$ defined in Section 2. In the following, we will drop the subscript α and write G^S_i, G^B_i instead. Suppose $z_i > z_j$. Then agent i's critical type $\omega_{\alpha,i}(z_i)$ interim expects that his ironed virtual type $\overline{\psi}_{\alpha,i}(\omega_{\alpha,i}(z_i), z_i) = z_i$ is greater than the ironed virtual type $\overline{\psi}_{\alpha,j}(x_j, z_j)$ of agent j with probability $G^B_j(z_i)$. Similarly, the critical type $\omega_{\alpha,j}(z_j)$ of agent j interim expects to have a higher ironed virtual type than agent i with probability $G^S_i(z_j)$. Note that G^S_i and G^B_i are strictly increasing, $G^S_i(z_i) < G^B_i(z_i)$ for all $z_i \in [\underline{z}, \overline{z}]$, $G^S_i(\underline{z}) = 0$, and $G^B_i(\overline{z}) = 1$.

Consider agent i and a vector of ironing parameters $\mathbf{z}$. Let the set of agents other than i that have an ironing parameter less than z_i be denoted by $\mathcal{L}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j < z_i\}$. Similarly, let the sets of agents with ironing parameter equal to and greater than z_i be denoted by $\mathcal{E}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j = z_i\}$ and $\mathcal{G}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j > z_i\}$, respectively. If $\mathcal{E}_i(\mathbf{z}) \neq \varnothing$ for some i, ties in terms of ironed virtual type have strictly positive probability.

Suppose ties are broken hierarchically according to h. For each agent i, let $\underline{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) < h(i)\}$ and $\overline{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) > h(i)\}$ denote the set of other agents with the same ironing parameter against whom agent i wins and loses ties, respectively. Hence, under hierarchy h, the expected share of critical type $\omega_{\alpha,i}(z_i)$ of agent i is

$$S_i(\omega_{\alpha,i}(z_i)) = p_i(\mathbf{z}, h) := \prod_{j \in \mathcal{L}_i(\mathbf{z}) \cup \underline{\mathcal{E}}_i(\mathbf{z},h)} G^B_j(z_i) \prod_{k \in \mathcal{G}_i(\mathbf{z}) \cup \overline{\mathcal{E}}_i(\mathbf{z},h)} G^S_k(z_i).$$

Let $\mathbf{p}(\mathbf{z}, h) := \big(p_1(\mathbf{z}, h), \dots, p_n(\mathbf{z}, h)\big)$. The outcome $\big(S^{\mathbf{z},\mathbf{a}}_1(\omega_{\alpha,1}(z_1)), \dots, S^{\mathbf{z},\mathbf{a}}_n(\omega_{\alpha,n}(z_n))\big)$ of every split hierarchical tie-breaking rule $\mathbf{a}$ is equal to a convex combination of $\mathbf{p}(\mathbf{z}, h)$ for different hierarchies $h \in H$. Consequently, the set of all possible expected shares given $\mathbf{z}$ is equal to the convex hull of the expected shares under fixed hierarchies, i.e.,

$$\Gamma_n(\mathbf{z}) = \text{Conv}(\{\mathbf{p}(\mathbf{z}, h) : h \in H\}).$$

Note that depending on $\mathbf{z}$, we may have $\mathbf{p}(\mathbf{z}, h_1) = \mathbf{p}(\mathbf{z}, h_2)$ for some $h_1 \neq h_2$. In particular, if all n elements of $\mathbf{z}$ are distinct, i.e., $\mathcal{E}_i(\mathbf{z}) = \varnothing$ for all i, then tie-breaking has no bite and all $\mathbf{p}(\mathbf{z}, h)$ coincide. In this case, $\Gamma_n(\mathbf{z})$ is a singleton. On the other hand, if $\mathbf{z}$ is such that $z_i = z$ for all i, i.e., $\mathcal{L}_i(\mathbf{z}) = \mathcal{G}_i(\mathbf{z}) = \varnothing$, then all $n!$ points $\mathbf{p}(\mathbf{z}, h)$ are distinct extreme points of the convex hull $\Gamma_n(\mathbf{z})$. In general, if $\mathbf{z}$ is such that its elements take $k \leq n$ distinct values $z^1, \dots, z^k$, then $\Gamma_n(\mathbf{z})$ is equal to the convex hull of $\prod_{l=1}^{k} m_l!$ distinct extreme points, where m_l denotes the number of agents i with $z_i = z^l$.

Lemma 3. *The correspondence Γ_n has the following properties:*

(i) For all $\mathbf{z} \in [\underline{z}, \overline{z}]^n$, $\Gamma_n(\mathbf{z})$ is nonempty and convex.

(ii) Γ_n is upper hemicontinuous.

Proof. (i) immediately follows from the discussion above. For (ii), we have to show that for any two sequences $\mathbf{z}^q \to \mathbf{z}$ and $\mathbf{y}^q \to \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, we have $\mathbf{y} \in \Gamma_n(\mathbf{z})$. Note that if $\mathbf{z}$ is such that all its components are distinct, $\Gamma_n(\mathbf{z})$ is a singleton that is continuous at $\mathbf{z}$. Moreover, if the sequence $\mathbf{z}^q \to \mathbf{z}$ is such that the sets of agents for which ironing parameters coincide stay the same over the whole sequence, $\Gamma_n(\mathbf{z}^q)$ and $\Gamma_n(\mathbf{z})$ are all equal to the convex hull of the same number of extreme points. Since these extreme points are continuous in $\mathbf{z}^q$, $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$ and $\mathbf{y}^q \to \mathbf{y}$ imply $\mathbf{y} \in \Gamma_n(\mathbf{z})$ in this case. Finally, suppose there are some i, j for which $z_i^q > z_j^q$ but $z_i = z_j$. Then, if $\mathbf{y}^q \to \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, there exists a hierarchical tie-breaking rule for $\mathbf{z}$ where $h(i) > h(j)$ for all i, j with $z_i^q > z_j^q$ and $z_i = z_j$ that induces $\mathbf{y}$. Hence, $\mathbf{y} \in \Gamma_n(\mathbf{z})$. □

Partitioning the domain of Γ_n In order to study properties of the image of Γ_n, it will prove useful to consider the following partition of the domain $[\underline{z}, \overline{z}]^n$. Define

$$\xi_n(z) := \Big\{\mathbf{z} \in [z, \overline{z}]^n : z_i = z \text{ for at least one } i \in \mathcal{N}\Big\}.$$

Note that $\xi_n(z) \cap \xi_n(z') = \varnothing$ for all $z \neq z'$. Moreover, $\bigcup_{z \in [\underline{z}, \overline{z}]} \xi_n(z) = [\underline{z}, \overline{z}]^n$. Consequently, ξ_n represents a partition of the domain of Γ_n. In addition, define

$$O_n(z) := \Gamma_n(\xi_n(z)).$$

Hence, the image of Γ_n can be written as $\Gamma_n([\underline{z}, \overline{z}]^n) = \bigcup_{z \in [\underline{z}, \overline{z}]} O_n(z)$. Below, we will determine properties of $O_n(z)$ and their implications for $\Gamma_n([\underline{z}, \overline{z}]^n)$.

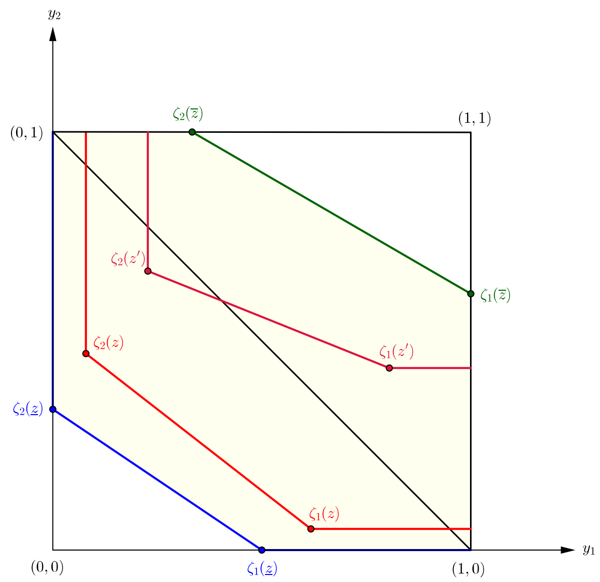


Figure 9: The image of Γ_2 and and its components for $\underline{z} < z < z' < \overline{z}$, with $O_2(\underline{z})$ in blue, $O_2(z)$ in light red, $O_2(z')$ in dark red, and $O_2(\overline{z})$ in green.

A.2 Proof of Properties 1 and 2 for $n = 2$

Suppose $n = 2$. There are only two possible hierarchies between two agents, i.e., $H = \{h_1, h_2\}$. Let h_1 (h_2) be the hierarchy where agent 1 (2) wins ties. Define $\boldsymbol{\zeta}_1(z) := \big(G_2^B(z), G_1^S(z)\big)$ and $\boldsymbol{\zeta}_2(z) := \big(G_2^S(z), G_1^B(z)\big)$. Hence, $\mathbf{p}(z, z, h_k) = \boldsymbol{\zeta}_k(z)$ for $k = 1, 2$. The general description of Γ_n in the preceding subsection implies

$$\Gamma_2(z_1, z_2) = \begin{cases} \big(G_2^B(z_1), G_1^S(z_2)\big) & \text{if } z_1 > z_2, \\ \text{Conv}\left(\left\{\boldsymbol{\zeta}_1(z), \boldsymbol{\zeta}_2(z)\right\}\right) & \text{if } z_1 = z_2 = z, \\ \big(G_2^S(z_1), G_1^B(z_2)\big) & \text{if } z_1 < z_2. \end{cases}$$

Suppose $z_1 = z_2 = z$. Geometrically, $\Gamma_2(z, z)$ is equal to all the points on the line segment from $\boldsymbol{\zeta}_1(z)$ to $\boldsymbol{\zeta}_2(z)$, i.e., all points in $\left\{a\boldsymbol{\zeta}_1(z) + (1-a)\boldsymbol{\zeta}_2(z) : a \in [0,1]\right\}$, where a is the share allocated to agent 1 in case of a tie (i.e., according to hierarchy h_1).

Now consider $O_2(z) = \Gamma_2(\xi_2(z))$ for some $z \in [\underline{z}, \overline{z}]$. In Figure 9, $O_2(z)$ is represented by the polygonal chain in light red. Geometrically, $O_2(z)$ consists of the line segment $\Gamma_2(z, z)$ with two line segments attached to its endpoints: a vertical line segment from $\boldsymbol{\zeta}_1(z)$ to $(1, G_1^S(z))$ that represents $\Gamma_2(z_1, z)$ for all $z_1 \in (z, \overline{z}]$ and a horizontal line segment from $\boldsymbol{\zeta}_2(z)$ to $(G_2^S(z), 1)$ that represents $\Gamma_2(z, z_2)$ for all $z_2 \in (z, \overline{z}]$.

Observe that both coordinates of the vertices $\boldsymbol{\zeta}_1(z)$ and $\boldsymbol{\zeta}_2(z)$ are continuous and strictly increasing in z. Hence, for $z' > z$, $O_2(z') \cap O_2(z) = \varnothing$ and $O_2(z')$ is further away from the origin than $O_2(z)$ (cf. the dark red line in Figure 9). Put differently, O_2 has the following monotonicity property: If $z' > z$, then for all $\mathbf{y}' \in O_2(z')$ and $\mathbf{y} \in O_2(z)$ we have $y_i' > y_i$ for at least one i.

Hence, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \overline{z}]^2)$, there is a unique z such that $\mathbf{y} \in O_2(z)$. Moreover, note that for each $\mathbf{y} \in O_2(z)$ there is a unique point $(z_1, z_2) \in \xi_2(z)$ such that $\mathbf{y} \in \Gamma_2(z_1, z_2)$. Consequently, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \overline{z}]^2)$ there is a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^2$ such that $\mathbf{y} \in \Gamma_2(\mathbf{z})$, i.e., *Property 1* holds for $n = 2$.

Consider $O_2(\underline{z})$ and note that $\boldsymbol{\zeta}_1(\underline{z}) = \left(G_2^B(\underline{z}), 0\right)$ and $\boldsymbol{\zeta}_2(\underline{z}) = \left(0, G_1^B(\underline{z})\right)$. Hence, the points $\mathbf{y} \in \Gamma_2(\underline{z}, \underline{z})$ all lie below the simplex Δ^1, which is represented by the black line segment from $(1, 0)$ to $(0, 1)$ in Figure 9. Moreover, the vertical and horizontal parts of $O_2(\underline{z})$ intersect with the simplex exactly at its boundary since $\left(1, G_1^S(\underline{z})\right) = (1, 0)$ and $\left(G_2^S(\underline{z}), 1\right) = (0, 1)$, respectively.

Let us increase z. For z small enough, the line segment $\Gamma_2(z, z)$ still lies below the simplex such that the vertical and horizontal part of $O_2(z)$ intersect with the simplex since the endpoints $(1, G_1^S(z))$ and $(G_2^S(z), 1)$ of $O_2(z)$ are above and to the left of the simplex for all $z > \underline{z}$. As z increases, the two intersection points move inwards on the simplex. As z becomes large enough, one of the two vertices $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ crosses the simplex, such that one intersection point lies in $\Gamma_2(z, z)$. The two intersection points approach each other until they coincide when the second vertex also crosses the simplex. Finally, for z sufficiently close to $\overline{z}$, both $\boldsymbol{\zeta}_1(z)$ and $\boldsymbol{\zeta}_2(z)$ and therefore the entire polygonal chain $O_2(z)$ lie above the simplex. To see this, note that $\boldsymbol{\zeta}_1(\overline{z}) = \left(1, G_1^S(\overline{z})\right)$ and $\boldsymbol{\zeta}_2(\overline{z}) = \left(G_2^S(\overline{z}), 1\right)$.

We have just shown that for every $\mathbf{y} \in \Delta^1$, there is a z such that $\mathbf{y} \in O_2(z)$. Consequently, $\Delta^1 \subset \Gamma_2([\underline{z}, \overline{z}]^2) = \bigcup_{z \in [\underline{z}, \overline{z}]} O_2(z)$, i.e., *Property 2* holds for $n = 2$. In Figure 9, $\Gamma_2([\underline{z}, \overline{z}]^2)$ is the yellow area between $O_2(\underline{z})$ and $O_2(\overline{z})$, representing a hexagon.

A.3 Proof of Properties 1 and 2 for $n > 2$

In the following, we will extend the approach of the previous subsection to $n > 2$. Characterizing O_n and Γ_n turns out to be significantly more complex in this case. To handle this complexity, we will first uncover the underlying recursive structure of Γ_n: one can construct Γ_n using modified versions of Γ_m for $m < n$. Exploiting this recursive structure, we will show that *Property 1* and *Property 2* hold for n if they hold for all $m < n$. Using $n = 2$ as the base case, the two properties then hold by induction for all n.

Suppose $z_1 = z_2 = \cdots = z_n = z$ and consider $\Gamma_n(z, \ldots, z) = \mathrm{Conv}(\{\mathbf{p}(z, \ldots, z, h) :$

$h \in H\}$). For each of the $n!$ different hierarchies $h \in H$,

$$\mathbf{p}(z,\dots,z,h) = \Bigg(\overbrace{\prod_{j\in\underline{\mathcal{E}}_1(h)} G_j^B(z) \prod_{k\in\overline{\mathcal{E}}_1(h)} G_k^S(z)}^{=p_1(z,\dots,z,h)},\dots,\overbrace{\prod_{j\in\underline{\mathcal{E}}_n(h)} G_j^B(z) \prod_{k\in\overline{\mathcal{E}}_n(h)} G_k^S(z)}^{=p_n(z,\dots,z,h)}\Bigg)$$

where we have simplified the notation by writing $\overline{\mathcal{E}}_i(h)$ instead of $\overline{\mathcal{E}}_i(z,\dots,z,h)$. Note that if $z > \underline{z}$, each $h \in H$ yields a distinct $\mathbf{p}(z,\dots,z,h)$. It can be shown that all points $\mathbf{p}(z,\dots,z,h)$ lie in the same $(n-1)$-dimensional hyperplane: For all $h \in H$,

$$\mathbf{p}(z,\dots,z,h) \in \Big\{\mathbf{y} \in \mathbb{R}^n : \sum_{i\in\mathcal{N}} \Big(G_i^B(z) - G_i^S(z)\Big)y_i = \prod_{j\in\mathcal{N}} G_j^B(z) - \prod_{j\in\mathcal{N}} G_j^S(z).\Big\}$$

Consequently, $\Gamma_n(z,\dots,z)$ is a $(n-1)$-dimensional convex polytope (in the hyperplane defined above) with vertices $\{\mathbf{p}(z,\dots,z,h) : h \in H\}$. Each vertex is connected to $n-1$ other vertices through an edge.

Now consider a nonempty subset of agents $\mathcal{K} \subset \mathcal{N}$ and denote its complement by $\mathcal{K}' := \mathcal{N} \setminus \mathcal{K}$. Define the set of hierarchies $H_{\mathcal{K}} \subset H$ such that for all $h \in H_{\mathcal{K}}$, we have $h(i) > h(j)$ for all $i \in \mathcal{K}$ and $j \in \mathcal{K}'$. If ties are broken based only on hierarchies in $H_{\mathcal{K}}$, agents in $\mathcal{K}$ always win ties against agents in $\mathcal{K}'$. The $(n-2)$-dimensional polytope $\mathrm{Conv}(\{\mathbf{p}(z,\dots,z,h) : h \in H_{\mathcal{K}}\})$ is a facet (i.e., an $(n-2)$-face) of the $(n-1)$-dimensional polytope $\Gamma_n(z,\dots,z)$. The boundary of $\Gamma_n(z,\dots,z)$ consists of $2^n - 2$ such facets, one for each possible nonempty $\mathcal{K} \subset \mathcal{N}$.[26]

Example with three agents In Subsection A.2 we have seen that $\Gamma_2(z,z)$ is a line segment. Assuming $n = 3$, there are 6 possible hierarchies, i.e., $H = \{h_1,\dots,h_6\}$. Hence, $\Gamma_3(z,z,z)$ is a hexagon (with opposite sides parallel). Let $\boldsymbol{\zeta}_l := \mathbf{p}(z,z,z,h_l)$ and suppose the hierarchies are enumerated in such a way that

$$\begin{aligned}
\boldsymbol{\zeta}_1 &= \Big(G_2^B(z)G_3^B(z), G_1^S(z)G_3^B(z), G_1^S(z)G_2^S(z)\Big), \boldsymbol{\zeta}_2 = \Big(G_2^B(z)G_3^B(z), G_1^S(z)G_3^S(z), G_1^S(z)G_2^B(z)\Big),\\
\boldsymbol{\zeta}_3 &= \Big(G_2^S(z)G_3^B(z), G_1^B(z)G_3^B(z), G_1^S(z)G_2^S(z)\Big), \boldsymbol{\zeta}_4 = \Big(G_2^S(z)G_3^S(z), G_1^B(z)G_3^B(z), G_1^B(z)G_2^S(z)\Big),\\
\boldsymbol{\zeta}_5 &= \Big(G_2^B(z)G_3^S(z), G_1^S(z)G_3^S(z), G_1^B(z)G_2^B(z)\Big), \boldsymbol{\zeta}_6 = \Big(G_2^S(z)G_3^S(z), G_1^B(z)G_3^S(z), G_1^B(z)G_2^B(z)\Big).
\end{aligned}$$

For example, $h_1(1) > h_1(2) > h_1(3)$ and $h_2(1) > h_2(3) > h_2(2)$. As shown in Figure 10, $\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_6$ are the vertices of the hexagon $\Gamma_3(z,z,z)$. The six edges $\overline{\boldsymbol{\zeta}_1\boldsymbol{\zeta}_3}$, $\overline{\boldsymbol{\zeta}_3\boldsymbol{\zeta}_4}$, $\overline{\boldsymbol{\zeta}_4\boldsymbol{\zeta}_6}$, $\overline{\boldsymbol{\zeta}_6\boldsymbol{\zeta}_5}$, $\overline{\boldsymbol{\zeta}_5\boldsymbol{\zeta}_2}$, and $\overline{\boldsymbol{\zeta}_2\boldsymbol{\zeta}_1}$ correspond to tie-breaking using $H_{\{1,2\}}$, $H_{\{2\}}$, $H_{\{2,3\}}$, $H_{\{3\}}$, $H_{\{1,3\}}$,

[26] There are $\binom{n}{k}$ facets where $|\mathcal{K}| = k$, each having $k!(n-k)!$ vertices.

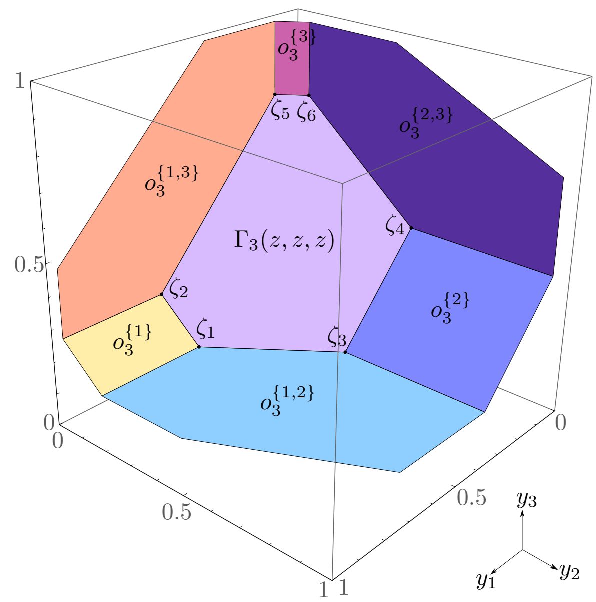


Figure 10: $O_3(z)$ and its components.

and $H_{\{1\}}$, respectively.[27]

Modified Γ_n correspondences and auxiliary definitions Below, we will use the following two modified versions of Γ_n. Let $\mathcal{M} = \{j_1, j_2, \ldots, j_m\} \subseteq \mathcal{N}$ be a subset of $m \geq 2$ agents. First, we denote by $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in $\mathcal{M}$ with modified virtual type distributions

$$\hat{G}_i^J(z) := G_i^J(z) \left(\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^B(z) \right)^{\frac{1}{m-1}} \qquad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Note that all the properties of virtual type distributions G_i^J carry over to modified virtual type distributions $\hat{G}_i^J$. In particular, $\hat{G}_i^B(z) > \hat{G}_i^S(z)$ for all $z \in [\underline{z}, \overline{z}]$, $\hat{G}_i^B(\overline{z}) = 1$, and $\hat{G}_i^S(\underline{z}) = 0$. Hence all results for Γ_m extend to $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$.

Second, we denote by $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in $\mathcal{M}$ with modified virtual type distributions

$$\check{G}_i^J(z) := G_i^J(z) \left(\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^S(z) \right)^{\frac{1}{m-1}} \qquad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Most properties of G_i^J carry over to their modified versions $\check{G}_i^J$, including $\check{G}_i^B(z) > \check{G}_i^S(z)$

[27]For $n = 4$, $\Gamma_4(z, z, z, z)$ is a truncated octahedron. In general, $\Gamma_n(z, \ldots, z)$ is reminiscent of a *permutahedron* (see, e.g., Ziegler, 1995), but its facets exhibit less symmetry (unless $F_i = F$ for all i).

for all $z \in (\underline{z}, \overline{z}]$ and $\check{G}_i^S(\underline{z}) = 0$. The only differences are $\check{G}_i^B(\overline{z}) < 1$, and $\check{G}_i^B(\underline{z}) = 0$. Again, all results for Γ_m extend to $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$, except for those relying on $\check{G}_i^B(\overline{z}) = 1$ or $\check{G}_i^B(\underline{z}) > 0$. In particular, note that $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}(z, \ldots, z)$ is equivalent to $\Gamma_m(z, \ldots, z)$ multiplied by the scalar $\prod_{k \in \mathcal{N} \setminus \mathcal{M}} G_k^S(z)$ (except for the m agents potentially being labeled differently).

We will also make use of the following auxiliary definitions for one-agent partnerships where $\mathcal{M}$ is a singleton: $\hat{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i \in \mathcal{N} \setminus j} G_i^B(z)$ and $\check{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i \in \mathcal{N} \setminus j} G_i^S(z)$ for all $z \in [\underline{z}, \overline{z}]$.

Recursive structure of O_n Let us now study $O_n(z) = \Gamma_n(\xi_n(z))$. Define

$$\xi_n^{\mathcal{K}}(z) := \Big\{ \mathbf{z} \in [z, \overline{z}]^n : z_i > z \text{ for } i \in \mathcal{K} \text{ and } z_j = z \text{ for } j \in \mathcal{K}' \Big\} \quad \text{for all } \mathcal{K} \subset \mathcal{N},$$

yielding a partition of $\xi_n(z)$ into $2^n - 1$ sets. Hence, $O_n(z) = \bigcup_{\mathcal{K} \subset \mathcal{N}} \Gamma_n\Big(\xi_n^{\mathcal{K}}(z)\Big)$.

Consider a specific $\mathcal{K} \subset \mathcal{N}$ and suppose $z_i > z$ for $i \in \mathcal{K}$ and $z_j = z$ for $j \in \mathcal{K}'$. Then, we can treat agents in $\mathcal{K}$ separately from agents in $\mathcal{K}'$. For the former, their critical type's expected share is as in a partnership among $k := |\mathcal{K}|$ agents with modified virtual type distributions $\hat{G}_i^J$ as defined above. For the latter, expected shares are as in $\Gamma_{n-k}(z, \ldots, z)$ but multiplied by the scalar $\prod_{i \in \mathcal{K}} G_i^S(z)$, i.e., as in a partnership with $n - k$ agents and modified virtual type distributions $\check{G}_i^J$. Given $\mathbf{y} \in [0,1]^n$, define $\mathbf{y}_{\mathcal{K}} := (y_{i_1}, y_{i_2}, \ldots, y_{i_k})$ for $\mathcal{K} = \{i_1, i_2, \ldots, i_k\}$ and $\mathbf{y}_{\mathcal{K}'} := (y_{j_1}, y_{j_2}, \ldots, y_{j_{n-k}})$ for $\mathcal{K}' = \{j_1, j_2, \ldots, j_{n-k}\}$. Hence, the closure of $\Gamma_n\Big(\xi_n^{\mathcal{K}}(z)\Big)$ is

$$o_n^{\mathcal{K}}(z) := \Big\{ \mathbf{y} \in [0,1]^n : \mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}\Big([z, \overline{z}]^k\Big) \text{ and } \mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z, \ldots, z) \Big\}.$$

Note that $\Gamma_{n-k}(z, \ldots, z)$ is an $(n - k - 1)$-dimensional convex polytope. If, in addition, $\Gamma_m([z, \overline{z}]^m)$ is an m-dimensional convex polytope for all $m < n$ (as we have already shown for $m = 2$ above), then $o_n^{\mathcal{K}}(z)$ is an $(n - 1)$-dimensional convex polytope for all $\mathcal{K}$.

With the definition above, $O_n(z) = \bigcup_{\mathcal{K} \subset \mathcal{N}} o_n^{\mathcal{K}}(z)$. Note that $o_n^{\varnothing}(z) = \Gamma_n(z, \ldots, z)$. Consequently, $O_n(z)$ is a polytopal complex that consists of $2^n - 1$ polytopes of dimension $(n - 1)$: $\Gamma_n(z, \ldots, z)$ with a polytope $o_n^{\mathcal{K}}(z)$ with nonempty $\mathcal{K}$ attached to each of its $2^n - 2$ facets.

Example with three agents (continued) $O_3(z)$ consists of the hexagon $\Gamma_3(z, z, z)$ with one polygon attached to each of its six edges, as shown in Figure 10. Those polygons can be divided into two groups: $o_3^{\{1\}}(z)$, $o_3^{\{2\}}(z)$, and $o_3^{\{3\}}(z)$ each represent a convex quadrilateral whereas $o_3^{\{1,2\}}(z)$, $o_3^{\{1,3\}}(z)$, and $o_3^{\{2,3\}}(z)$ are hexagons. For example,

$$o_3^{\{1\}}(z) = \Big\{\mathbf{y} \in [0,1]^3 : y_1 \in \hat{\Gamma}_1^{\{1\}:\mathcal{N}}\big([z,\overline{z}]\big) \text{ and } (y_2,y_3) \in \check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z,z)\Big\}.$$

Since both $\hat{\Gamma}_1^{\{1\}:\mathcal{N}}\big([z,\overline{z}]\big)$ and $\check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z,z)$ are line segments, $o_3^{\{1\}}(z)$ is a convex quadrilateral, sharing the edge $\overline{\boldsymbol{\zeta}_2\boldsymbol{\zeta}_1}$ with the hexagon $\Gamma_3(z,z,z)$. Moreover,

$$o_3^{\{1,2\}}(z) = \Big\{\mathbf{y} \in [0,1]^3 : (y_1,y_2) \in \hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}\big([z,\overline{z}]^2\big) \text{ and } y_3 = \check{\Gamma}_1^{\{3\}:\mathcal{N}}(z)\Big\}.$$

Note that y_3 is constant whereas $\hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}\big([z,\overline{z}]^2\big)$ is a hexagon, which follows from Subsection A.2 (cf. Figure 9). Hence, $o_3^{\{1,2\}}(z)$ is also a hexagon, sharing the edge $\overline{\boldsymbol{\zeta}_1\boldsymbol{\zeta}_3}$ with the hexagon $\Gamma_3(z,z,z)$.

Monotonicity of O_n Observe that all coordinates of each $\mathbf{p}(z,\dots,z,h)$ are continuous and strictly increasing in z. Hence, if $\hat{z} > z$, then for all $\hat{\mathbf{y}} \in \Gamma_n(\hat{z},\dots,\hat{z})$ and $\mathbf{y} \in \Gamma_n(z,\dots,z)$ we have $\hat{y}_i > y_i$ for at least one i. The following lemma shows that the monotonicity property of $\Gamma_n(z,\dots,z)$ extends to $O_n(z)$.

Lemma 4. *If $\hat{z} > z$, then for all $\hat{\mathbf{y}} \in O_n(\hat{z})$ and $\mathbf{y} \in O_n(z)$, $\hat{y}_i > y_i$ for at least one i.*

Proof. We will show that $\hat{y}_i > y_i$ for at least one i for all $\mathcal{K},\mathcal{M} \subset \mathcal{N}$ and $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z}), \mathbf{y} \in o_n^{\mathcal{K}}(z)$.

Note that each $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ corresponds to a $\hat{\mathbf{z}} \in [\hat{z},\overline{z}]^n$ and a tie-breaking rule. Now, consider the $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ that is obtained when breaking ties among agents in $\mathcal{K}'$ in such a way that the same rule as for $\hat{\mathbf{y}}$ is applied for all $j,l \in \mathcal{K}'$ where $\hat{z}_j = \hat{z}_l$, whereas j wins against l for all $j,l \in \mathcal{K}'$ where $\hat{z}_j > \hat{z}_l$. This tie-breaking implies $\hat{y}_j > \tilde{y}_j$ for all $j \in \mathcal{K}' \cap \mathcal{M}$ since $p_j(\hat{\mathbf{z}},h) > p_j(\hat{z},\dots,\hat{z},h)$ for all relevant hierarchies h. Moreover $\hat{y}_l \geq \tilde{y}_l$ for all $l \in \mathcal{K}' \cap \mathcal{M}'$. Hence, we conclude that for all $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ there is a $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ such that $\hat{y}_i \geq \tilde{y}_i$ for all $i \in \mathcal{K}'$.

Since $\hat{z} > z$, there is for all $\tilde{\mathbf{y}}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ and $\mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z,\dots,z)$ at least one $i \in \mathcal{K}'$ such that $\tilde{y}_i > y_i$. Combining this with the conclusion of the preceding paragraph implies that for all $\hat{\mathbf{y}} \in o_n^{\mathcal{M}}(\hat{z})$ and $\mathbf{y} \in o_n^{\mathcal{K}}(z)$ there is at least one $i \in \mathcal{K}'$ such that $\hat{y}_i \geq \tilde{y}_i > y_i$. □

For the three-agent example displayed in Figure 10, Lemma 4 implies that $O_3(z)$ moves towards the observer as we increase z. See also Figure 11 below that depicts $O_3(z)$ for four different values for z.

Induction step for Property 1 Monotonicity of O_n implies that for each $\mathbf{y} \in \Gamma_n([\underline{z},\overline{z}]^n) = \bigcup_{z\in[\underline{z},\overline{z}]} O_n(z)$ there is a unique z such that $\mathbf{y} \in O_n(z)$.

Lemma 5. *If Property 1 holds for all Γ_m with $m < n$, then Property 1 holds for Γ_n.*

Proof. Lemma 4 implies that for every $\mathbf{y} \in \Gamma_n([\underline{z},\overline{z}]^n)$ there is a unique z such that $\mathbf{y} \in O_n(z)$.

We will next show that for every $\mathbf{y} \in O_n(z)$, there is a unique $\mathcal{K} \subset \mathcal{N}$ such that $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$. Consider $\mathcal{K}, \mathcal{M} \subset \mathcal{N}$ such that $\mathcal{K} \neq \mathcal{M}$. Without loss of generality, suppose $\mathcal{K} \cap \mathcal{M}' \neq \varnothing$. Then, for all $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$ and $\tilde{\mathbf{y}} \in \Gamma_n(\xi_n^{\mathcal{M}}(z))$, $y_i > \tilde{y}_i$ for at least one $i \in \mathcal{K} \cap \mathcal{M}'$. To see this, consider the corresponding $\mathbf{z} \in \xi_n^{\mathcal{K}}(z)$ and $\tilde{\mathbf{z}} \in \xi_n^{\mathcal{M}}(z)$. For $i \in \mathcal{K} \cap \mathcal{M}'$ and $j \in \mathcal{K}'$, we have $z_i > z_j = z$ but $\tilde{z}_i = z \leq \tilde{z}_j$. Hence, in the first case the critical type of agent i has a strictly higher winning probability against agents in $\mathcal{K}'$ than in the second case. The same is true for $j \in \mathcal{K} \cap \mathcal{M}$, since $z_i > z$ whereas $\tilde{z}_i = z < \tilde{z}_j$. Finally, the winning probability of agent i's critical type against other agents in $\mathcal{K} \cap \mathcal{M}'$ cannot be lower for all $i \in \mathcal{K} \cap \mathcal{M}'$ when considering $\mathbf{z} \in \xi_n^{\mathcal{K}}(z)$ than when considering $\tilde{\mathbf{z}} \in \xi_n^{\mathcal{M}}(z)$. Consequently, $y_i > \tilde{y}_i$ for at least one i.

So far we have shown that for every $\mathbf{y} \in \Gamma_n([\underline{z},\overline{z}]^n)$, there are unique $z, \mathcal{K}$ such that $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$. This already partially pins down $\mathbf{z}$: for all $i \in \mathcal{K}'$, we have $z_i = z$. Moreover, $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$ implies $\mathbf{y} \in o_n^{\mathcal{K}}(z)$ and therefore $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}\Big([z,\overline{z}]^k\Big)$. If *Property 1* holds for $k < n$, there is a unique $\mathbf{z}_{\mathcal{K}}$ such that $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}\Big(\mathbf{z}_{\mathcal{K}}\Big)$. This pins down z_i also for $i \in \mathcal{K}$. □

Convexity of $\Gamma_n\Big([z,\overline{z}]^n\Big)$ Suppose $\Gamma_m\Big([z,\overline{z}]^m\Big)$ is a convex polytope for all $m < n$. As observed above, this implies that $O_n(z)$ is a polytopal complex consisting of $2^n - 1$ convex polytopes $o_n^{\mathcal{K}}$ of dimension $n-1$, one for each $\mathcal{K} \subset \mathcal{N}$. If $\mathcal{K} \cap \mathcal{M} \neq \varnothing$, then the two polytopes $o_n^{\mathcal{K}}$ and $o_n^{\mathcal{M}}$ are adjacent, i.e., they share a facet (of dimension $n-2$). Let the boundary of the polytopal complex $O_n(z)$ be defined as all the facets of each polytope $o_n^{\mathcal{K}}$ that are not shared with some other polytope $o_n^{\mathcal{M}}$, where $\mathcal{K} \neq \mathcal{M}$. Each point $\mathbf{y} \in \Gamma_n(\mathbf{z})$ on the boundary of $O_n(z)$ corresponds to a $\mathbf{z}$ where, for some $\mathcal{K} \subset \mathcal{N}$, $z_i = \overline{z}$ for $i \in \mathcal{K}$ whereas $z_j = z$ for $j \in \mathcal{K}'$.

In a similar manner as we constructed O_n above, define

$$Q_n(z) := \Gamma_n\Big(\Big\{\mathbf{z} \in [z,\overline{z}]^n : z_i = \overline{z} \text{ for at least one } i \in \mathcal{N}\Big\}\Big) = \bigcup_{\mathcal{K}\subset\mathcal{N}} q_n^{\mathcal{K}}(z)$$

where

$$q_n^{\mathcal{K}}(z) := \Big\{\mathbf{y} \in [0,1]^n : \mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}(\overline{z},\ldots,\overline{z}) \text{ and } \mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}\Big([z,\overline{z}]^{n-k}\Big)\Big\}.$$

$Q_n(z)$ represents the image under Γ_n of the set of all $\mathbf{z}$ where $z_i \geq z$ for all i and $z_i = \overline{z}$ for at least one i. Observe that $Q_n(z)$ contains all the boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in [z, \overline{z}]$. Moreover, $Q_n(\overline{z}) = O_n(\overline{z})$.

Lemma 6. $\Gamma_n\Big([z, \overline{z}]^n\Big)$ *is an n-dimensional convex polytope for all $z < \overline{z}$. The boundary of this polytope is $O_n(z) \cup Q_n(z)$.*

Proof. From Subsection A.2 we know that $\Gamma_2\Big([z, \overline{z}]^2\Big)$ is a hexagon. We will now show that if $\Gamma_m\Big([z, \overline{z}]^m\Big)$ is a convex polytope for all $m < n$, then $\Gamma_n\Big([z, \overline{z}]^n\Big)$ is a convex polytope. Consequently, the first statement in the lemma follows by induction.

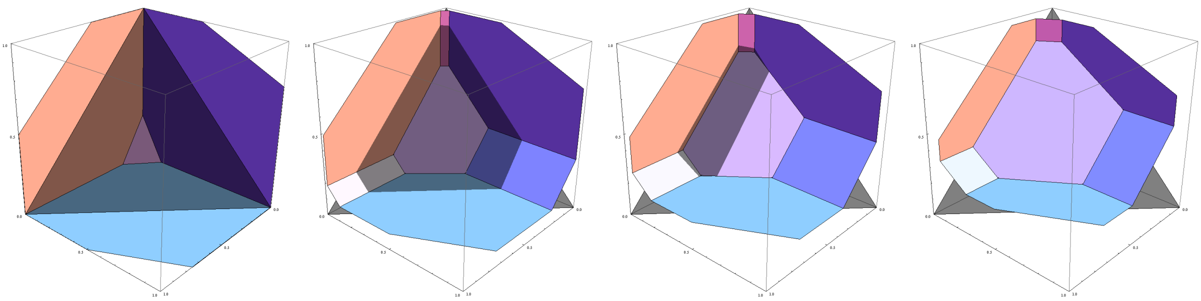
Suppose $\Gamma_m\Big([z, \overline{z}]^m\Big)$ is a convex polytope for all $m < n$ and recall that $\Gamma_n\Big([z, \overline{z}]^n\Big) = \bigcup_{\tilde{z} \in [z, \overline{z}]} O_n(\tilde{z})$. As derived above, $O_n(z)$ is a polytopal complex. As all coordinates of the extreme points of $\Gamma_n(z, \ldots, z)$ are continuous and strictly increasing in z, Lemma 4 implies that $O_n(z)$ continuously moves further away from the origin as z increases. Hence, $O_n(z)$ is part of the boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$.

In addition to $O_n(z)$, all boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in (z, \overline{z})$ are also part of the boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$ whereas all interior points of $O_n(\tilde{z})$ are in the interior of $\Gamma_n\Big([z, \overline{z}]^n\Big)$. Lastly, note that $O_n(\overline{z})$ consists of only one convex polytope (namely $\Gamma_n(\overline{z}, \ldots, \overline{z})$) and that all its points are part of the boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$.

$Q_n(z)$ represents all points on the boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$ described in the preceding paragraph, i.e., boundary points that are not in $O_n(z)$. Consequently, $O_n(z) \cup Q_n(z)$ represents the entire boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$. Like $O_n(z)$, $Q_n(z)$ is also a polytopal complex that consists of $2^n - 1$ convex polytopes of dimension $n - 1$. The boundary of $\Gamma_n\Big([z, \overline{z}]^n\Big)$ therefore consists of $2^{n+1} - 2$ convex polytopes ($o_n^{\mathcal{K}}(z)$ and $q_n^{\mathcal{K}}(z)$ for all $\mathcal{K} \subset \mathcal{N}$), making $\Gamma_n\Big([z, \overline{z}]^n\Big)$ an n-dimensional polytope with $2^{n+1} - 2$ facets.

Recall that for all $z < \overline{z}$, $O_n(z)$ consists of $\Gamma_n(z, \ldots, z)$ with a $o_n^{\mathcal{K}}(z)$ attached to each facet. The points in each $o_n^{\mathcal{K}}(z)$ are further away from the origin than the points on the corresponding facet of $\Gamma_n(z, \ldots, z)$. Because of the monotonicity and continuity properties of $O_n(z)$, for all $\mathbf{y} \in \operatorname{Conv}\Big(O_n(z)\Big)$ such that $\mathbf{y} \notin O_n(z)$, there is a $\tilde{z} \in (z, \overline{z}]$ such that $\mathbf{y} \in O_n(\tilde{z})$. Hence, the polytope $\Gamma_n\Big([z, \overline{z}]^n\Big) = \bigcup_{\tilde{z} \in [z, \overline{z}]} O_n(\tilde{z})$ is convex. □

Induction step for Property 2 Consider $O_n(\underline{z})$. This represents a special case since $\Gamma_n(\underline{z}, \ldots, \underline{z})$ is a general $(n-1)$-simplex with only n vertices rather than a polytope with $n!$ vertices. In particular, note that for each vertex $\mathbf{p}(\underline{z}, \ldots, \underline{z}, h) = (p_1, \ldots, p_n)$, $p_i \in (0, 1)$ for one i whereas $p_j = 0$ for all $j \neq i$, resulting in only n distinct vertices. Since $\sum_{i=1}^n p_i < 1$, the general simplex $\Gamma_n(\underline{z}, \ldots, \underline{z})$ does not intersect with standard simplex



(a) $O_3(\underline{z})$ (b) $O_3(z)$ (c) $O_3(z')$ (d) $O_3(z'')$

Figure 11: Increasing z in the three-player example. O_3 for some $\underline{z} < z < z' < z'' < \overline{z}$ and the simplex Δ^2 (semitransparent black triangle).

Δ^{n-1}: the former lies closer to the origin than the latter.[28]

It follows that $O_n(\underline{z})$ consists of only $n+1$ polytopes of dimension $(n-1)$: The general simplex $\Gamma_n(\underline{z},\ldots,\underline{z})$ with a polytope o_n^i attached to each of its n facets (each corresponding to a general $n-2$-simplex), where, for each $i \in \mathcal{N}$,

$$o_n^i := \left\{\mathbf{y} \in [0,1]^n : \mathbf{y}_{\mathcal{N}\setminus i} \in \hat{\Gamma}_{n-1}^{\mathcal{N}\setminus i:\mathcal{N}}\left([\underline{z},\overline{z}]^{n-1}\right) \text{ and } y_i = 0\right\}.$$

Lemma 7. *If Property 2 holds for all Γ_m with $m < n$, then $O_n(\underline{z})$ contains the entire boundary (all n facets) of Δ^{n-1}.*

Proof. $O_n(\underline{z})$ is the union of $\Gamma_n(\underline{z},\ldots,\underline{z})$ and n polytopes o_n^i as defined above. *Property 2* for $m < n$ implies in particular $\Delta^{n-2} \subset \Gamma_{n-1}\left([\underline{z},\overline{z}]^{n-1}\right)$ and therefore $\Delta^{n-2} \subset \hat{\Gamma}_{n-1}^{\mathcal{N}\setminus i:\mathcal{N}}\left([\underline{z},\overline{z}]^{n-1}\right)$. Moreover, the n facets of Δ^{n-1} all correspond to one coordinate being set to zero, i.e., $\mathbf{y}_{\mathcal{N}\setminus i} \in \Delta^{n-2}$ and $y_i = 0$. □

Panel (a) of Figure 11 illustrates Lemma 7 in the three-agent example. It shows how $O_3(\underline{z})$ intersects with the boundary of the semitransparent black triangle that represents the simplex Δ^2. Figure 11 also conveys that as we increase z, the intersection of $O_2(z)$ with Δ^2 moves inward (Panels (b) and (c)) until the entire simplex has been covered and for all higher z, $O_2(z)$ does not intersect with Δ^2 (Panel (d)).[29] Hence, *Property 2* holds for Γ_3.

Using the convexity of $\Gamma_n\left([\underline{z},\overline{z}]^n\right)$, it is now straightforward to obtain the following lemma.

[28]In the three-agent example above, we obtain, for $z = \underline{z}$, $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2 = \left(G_2^B(\underline{z})G_3^B(\underline{z}),\ 0,\ 0\right)$, $\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_4 = \left(0,\ G_1^B(\underline{z})G_3^B(\underline{z}),\ 0\right)$, $\boldsymbol{\zeta}_5 = \boldsymbol{\zeta}_6 = \left(0,\ 0,\ G_1^B(\underline{z})G_2^B(\underline{z})\right)$.

[29]Let $\hat{z}$ be the smallest z such that $\sum_{i\in\mathcal{N}} p_i(z,\ldots,z,h) \geq 1$ for all $h \in H$. Similarly, let $\check{z}$ be the greatest z such that $\sum_{i\in\mathcal{N}} p_i(z,\ldots,z,h) \leq 1$ for all $h \in H$. Observe that $\underline{z} < \check{z} \leq \hat{z} < \overline{z}$ (with $\check{z} = \hat{z}$ if $F_i = F$ for all i). $O_n(z)$ intersects with Δ^{n-1} if and only if $z \leq \hat{z}$ whereas $\Gamma_n(z,\ldots,z)$ intersects with Δ^{n-1} if and only if $z \in [\check{z},\hat{z}]$. Panels (b), (c), and (d) of Figure 11 correspond to $z < \check{z} < z' < \hat{z} < z''$.

Lemma 8. *If Property 2 holds for all* Γ_m *with* $m < n$*, then Property 2 holds for* Γ_n*.*

Proof. If *Property 2* holds for all Γ_m with $m < n$, then, according to Lemma 7, $O_n(\underline{z})$ contains the entire boundary of Δ^{n-1}. By Lemma 6, $\Gamma_n\Big([\underline{z},\overline{z}]^n\Big)$ is convex and $O_n(\underline{z})$ is part of the boundary of $\Gamma_n\Big([\underline{z},\overline{z}]^n\Big)$. Consequently, the boundary of Δ^{n-1} being contained in the boundary of $\Gamma_n\Big([\underline{z},\overline{z}]^n\Big)$ implies *Property 2* for Γ_n. □

Final step As shown in Subsection A.2, *Property 1* and *Property 2* hold for $n = 2$. By induction, using Lemmata 5 and 8, *Property 1* and *Property 2* hold for all n.

Appendix B: Other Proofs

B.1 Proof of Lemma 1

The definition of U_i implies

$$W_\alpha(\mathbf{s},\mathbf{t}) = \sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})(s_i(\mathbf{X}) - r_i)] - \alpha \sum_{i\in\mathcal{N}} E[U_i(X_i)] + (1-\alpha) \sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})r_i]. \quad (20)$$

Using the fact that $\sum_{i\in\mathcal{N}}(s_i(\mathbf{X}) - r_i) = 0$, we get

$$\begin{aligned}\sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})\,(s_i(\mathbf{X}) - r_i)] &= \sum_{i\in\mathcal{N}} E\left[\left(X_i - \eta(X_i) + \sum_{j\in\mathcal{N}} \eta(X_j)\right)(s_i(\mathbf{X}) - r_i)\right] \\ &= \sum_{i\in\mathcal{N}} E\Big[(X_i - \eta(X_i))\,(S_i(X_i) - r_i)\Big]. \quad (21)\end{aligned}$$

Integrating (IC2) by parts, we obtain for all $\hat{x}_i \in [0,1]$

$$\begin{aligned}E[U_i(X_i)] &= U_i(\hat{x}_i) + \int_0^1 \int_{\hat{x}_i}^{x_i} (S_i(z) - r_i)dz f_i(x_i)dx_i \\ &= U_i(\hat{x}_i) - \int_0^{\hat{x}_i} F_i(z)(S_i(z) - r_i)dz + \int_{\hat{x}_i}^1 (1 - F_i(z))(S_i(z) - r_i)dz. \quad (22)\end{aligned}$$

Substituting (21) and (22) into (20) yields

$$\begin{aligned}W_\alpha(\mathbf{s},\mathbf{t}) = \sum_{i\in\mathcal{N}} \left(\int_0^{\hat{x}_i} \psi^S_{\alpha,i}(z)(S_i(z) - r_i)f_i(z)dz + \int_{\hat{x}_i}^1 \psi^B_{\alpha,i}(z)(S_i(z) - r_i)f_i(z)dz\right) \\ - \alpha \sum_{i\in\mathcal{N}} U_i(\hat{x}_i) + (1-\alpha) \sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})r_i]\end{aligned}$$

which, by the definitions of $\psi_{\alpha,i}(x_i,\hat{x}_i)$ and $\widetilde{W}_\alpha(\mathbf{s},\hat{\mathbf{x}})$, is equivalent to (3).

Consider $\hat{\mathbf{x}}, \boldsymbol{\omega} \in [0,1]^n$. From (3), we obtain

$$\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) - \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) = \alpha \sum_{i \in \mathcal{N}} \Big(U_i(\hat{x}_i) - U_i(\omega_i) \Big).$$

For all $\boldsymbol{\omega} \in \Omega(\mathbf{s})$ and $\hat{\mathbf{x}} \notin \Omega(\mathbf{s})$, we have $U_i(\hat{x}_i) \geq U_i(\omega_i)$ for all i, where the inequality is strict for at least one i, and therefore $\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) > \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega})$. Consequently, $\Omega(\mathbf{s}) = \arg\min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$. □

B.2 Proof of Proposition 1

Suppose agent 2 plays the candidate equilibrium strategy given in the proposition. We will show that it is a best response for agent 1 to also play this strategy.

First, consider the continuation game after both agents have chosen BUY. Agent 1 infers that $x_2 > \overline{\omega}$ and has to choose his strategy in the open ascending forward auction. Given that agent 2 follows the strategy to stay in the auction until the price reaches $v_2(x_2, x_2)$, agent 1's payoff from winning the auction is $v_1(x_1, x_2) - v_2(x_2, x_2)$, which is positive for $x_2 \leq x_1$. Hence, it is optimal for agent 1 to drop out at price $v_1(x_1, x_1)$. (If $x_1 < \overline{\omega}$, which is off the equilibrium path, 1 drops out immediately.) Essentially the same argument applies to the continuation game after both agents have chosen SELL: agent 1 chooses to drop out at price $v_1(x_1, x_1)$ in the open descending reverse auction.

Now, consider the first stage where agent 1 chooses between BUY, HOLD, and SELL:

- Conditional on $X_2 < \underline{\omega}$, i.e., 2 choosing SELL, agent 1's expected payoff is

  $$\tfrac{1}{2}\Big(E[v_1(x_1, X_2) | X_2 < \underline{\omega}] - p^B\Big) + E[\max\{v_2(X_2, X_2) - v_1(x_1, X_2), 0\} | X_2 < \underline{\omega}]$$

  if he chooses SELL and $\tfrac{1}{2}\Big(E[v_1(x_1, X_2) | X_2 < \underline{\omega}] - p^B\Big)$ if he chooses HOLD or BUY.

- Conditional on $X_2 \in [\underline{\omega}, \overline{\omega}]$, i.e., 2 choosing HOLD, agent 1's expected payoff is $\tfrac{1}{2}\Big(\hat{p}^S - E[v_1(x_1, X_2) | \underline{\omega} \leq X_2 \leq \overline{\omega}]\Big)$ if he chooses SELL, 0 if he chooses HOLD, and $\tfrac{1}{2}\Big(E[v_1(x_1, X_2) | \underline{\omega} \leq X_2 \leq \overline{\omega}] - \hat{p}^B\Big)$ if he chooses BUY.

- Conditional on $X_2 > \overline{\omega}$, i.e., 2 choosing BUY, agent 1's expected payoff is $\tfrac{1}{2}\Big(p^S - E[v_1(x_1, X_2) | X_2 > \overline{\omega}]\Big)$ if he chooses SELL or HOLD and

  $$\tfrac{1}{2}\Big(p^S - E[v_1(x_1, X_2) | X_2 > \overline{\omega}]\Big) + E[\max\{v_1(x_1, X_2) - v_2(X_2, X_2), 0\} | X_2 > \overline{\omega}]$$

  if he chooses BUY.

Consequently, in all three cases agent 1 finds it optimal to choose SELL if $x_1 < \underline{\omega}$, HOLD if $\underline{\omega} \le x_1 \le \overline{\omega}$, and BUY otherwise.

Note that the allocation induced in this equilibrium is equal to the optimal allocation of Theorem 1. Hence, by incentive compatibility, the induced payments are also pinned down up to a constant. Now, note that according to Theorem 1, the interim payment of types $x_1 \in [\underline{\omega}, \overline{\omega}]$ in an optimal dissolution mechanism is

$$\begin{aligned} T_1(x_1) &= E[v_1(x_1, X_2)(s_1(x_1, X_2) - \tfrac{1}{2})] \\ &= F(\underline{\omega})E[v_1(x_1, X_2)(1 - \tfrac{1}{2})|X_2 < \underline{\omega}] + (1 - F(\overline{\omega}))E[v_1(x_1, X_2)(-\tfrac{1}{2})|X_2 > \overline{\omega}] \\ &= F(\underline{\omega})\tfrac{1}{2}p^B - (1 - F(\overline{\omega}))\tfrac{1}{2}p^S, \end{aligned}$$

which coincides with the expected payment of those types in the indirect mechanism. □

B.3 Proof of Corollary 3

If $F_i = F$, then $\omega_{\alpha,i} = \omega_\alpha$ and $E[v_i(\omega_\alpha(z_i), \mathbf{X}_{-i})]$ is symmetric across agents. Theorem 2 implies that all optimal initial shares $\mathbf{r}^*$ are such that $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}^*)$ satisfies $z_i^* = z^*$ for all $i \in \mathcal{N}$. As we will show next, there is a unique such z^*.

Consider an ironed virtual type allocation rule $\mathbf{s}^{\mathbf{z},h}$ with $z_i = z$ for all i and hierarchical tie-breaking according to $h \in H$. Under such an allocation rule, agent i's critical type obtains the object if all agents j with $h(j) < h(i)$ have virtual valuations below z and all agents k with $h(k) > h(i)$ have virtual costs below z. Consequently, $S_i^{\mathbf{z},h}(\omega_\alpha(z)) = \left(G_\alpha^S(z)\right)^{n-h(i)}\left(G_\alpha^B(z)\right)^{h(i)-1}$. Each hierarchy $h' \neq h$ corresponds to a permutation of the components of $\left(S_1^{\mathbf{z},h}(\omega_\alpha(z)), \ldots, S_n^{\mathbf{z},h}(\omega_\alpha(z))\right)$. $\Gamma_n(z, \ldots, z)$ is the convex hull of the set of all permutations of $\left(S_1^{\mathbf{z},h}(\omega_\alpha(z)), \ldots, S_n^{\mathbf{z},h}(\omega_\alpha(z))\right)$. Since G_α^S, G_α^B are strictly increasing, there is a unique z^* such that $\sum_{i \in \mathcal{N}} \left(G_\alpha^S(z^*)\right)^{n-i}\left(G_\alpha^B(z^*)\right)^{i-1} = 1$. It follows that $\Gamma_n(z^*, \ldots, z^*) \subset \Delta^{n-1}$ whereas $\Gamma_n(z, \ldots, z) \cap \Delta^{n-1} = \varnothing$ for all $z \neq z^*$.

Rado's Theorem (Marshall, Olkin, and Arnold, 2011, p. 34) implies that $\mathbf{r} \in \Gamma_n(z^*, \ldots, z^*)$ is equivalent to $\mathbf{r} \prec \left(S_1^{\mathbf{z},h}(\omega_\alpha(z^*)), \ldots, S_n^{\mathbf{z},h}(\omega_\alpha(z^*))\right)$. For $h(i) = i$, the RHS is equal to $\mathbf{r}^\alpha$. □

B.4 Proof of Corollary 4

Solving (16) for r_i^*, we obtain $r_i^* = \prod_{j \neq i} F_j\Big(Y - \sum_{l \neq i} E[\eta(X_l)]\Big)$. Since the RHS vanishes if and only if $Y \le \sum_{l \neq i} E[\eta(X_l)]$, this expression also represents (17). Observe that Y is then uniquely pinned down by the condition

$$\sum_{i\in\mathcal{N}} r_i^* = 1 = \sum_{i\in\mathcal{N}} \prod_{j\neq i} F_j\Big(Y - \sum_{l\neq i} E[\eta(X_l)]\Big),$$

rendering the optimal ownership structure unique. Assuming $E[\eta(X_1)] \geq E[\eta(X_2)] \geq \cdots \geq E[\eta(X_n)]$, the expression $Y - \sum_{l\neq i} E[\eta(X_l)]$ is maximized at $i = 1$ for every Y. Hence, agent 1's optimal share is always strictly positive. Using $Y = \tilde{\omega}_{0,1}(r_1^*) + \sum_{l\neq 1} E[\eta(X_l)]$, we can express the condition on Y as a condition on r_1^* and obtain the characterization in the corollary. Finally, note that $\tilde{\omega}_{0,1}(r_1^*) - E[\eta(X_1)] + E[\eta(X_i)] \leq 0$ and therefore $r_i^* = 0$ for some i implies $\tilde{\omega}_{0,1}(r_1^*) - E[\eta(X_1)] + E[\eta(X_j)] \leq 0$ and $r_j^* = 0$ for all $j > i$. □

B.5 Proof of Proposition 2

Part (i): Note that $F_i(x) < F_j(x)$ implies $\tilde{\omega}_{0,i}(r) < \tilde{\omega}_{0,j}(r)$ for all $r \in (0,1)$. Hence, if $E[\eta(X_i)] \geq E[\eta(X_j)]$,

$$\tilde{\omega}_{0,i}(r) + \sum_{k\neq i} E[\eta(X_k)] < \tilde{\omega}_{0,j}(r) + \sum_{k\neq j} E[\eta(X_k)] \quad \text{for all } r \in (0,1).$$

First, suppose $r_i^* > 0$. Then either $r_j^* = 0$ or (16) implies

$$\tilde{\omega}_{0,i}(r_i^*) + \sum_{l\neq i} E[\eta(X_l)] = \tilde{\omega}_{0,i}(r_j^*) + \sum_{l\neq j} E[\eta(X_l)]$$

and therefore $r_i^* > r_j^*$. Moreover, $r_i^* = 1$ is impossible, as (16) and (17) would then result in the contradiction

$$1 \leq E[\eta(X_i)] - E[\eta(X_j)] = \int_0^1 \eta'(x)\Big(F_j(x) - F_i(x)\Big)dx < 1$$

because $\eta'(x) < 1$ for all x. Now, suppose $r_i^* = 0$. In this case, $r_j^* = 0$ is immediate from Corollary 4.

Part (ii): If $E[\eta(X_i)] \leq E[\eta(X_j)] - 1$, Corollary 4 implies $r_i^* = 0$ whereas r_j^* depends on the properties of the agents other than i and j. □

B.6 Proof of Proposition 3

To prove the proposition, we will make use of the following lemma.

Lemma 9. *For each $i \in \mathcal{N}$, $\omega_{\alpha,i}(\underline{z})$ is strictly increasing and $\omega_{\alpha,i}(\overline{z})$ strictly decreasing in α. Both $\omega_{\alpha,i}(\underline{z})$ and $\omega_{\alpha,i}(\overline{z})$ are continuous in α and $\omega_{\alpha,i}(\underline{z}) < E[X_i] < \omega_{\alpha,i}(\overline{z})$ for all $\alpha \in [0,1]$. Moreover, $\lim_{\alpha\to 0}\omega_{\alpha,i}(\underline{z}) = 0$ and $\lim_{\alpha\to 0}\omega_{\alpha,i}(\overline{z}) = 1$.*

Proof. First observe that

$$\alpha\omega_{\alpha,i}(z) = \int_{\psi^B_{\alpha,i}(0)}^{z} \Big(G^B_{\alpha,i}(y) - G^S_{\alpha,i}(y)\Big)dy$$
$$= \Big(G^B_{\alpha,i}(z) - G^S_{\alpha,i}(z)\Big)z - G^B_{\alpha,i}(z)E\Big[\psi^B_{\alpha,i}(X_i)\Big|\psi^B_{\alpha,i}(X_i) \le z\Big] + G^S_{\alpha,i}(z)E\Big[\psi^S_{\alpha,i}(X_i)\Big|\psi^S_{\alpha,i}(X_i) \le z\Big].$$

Using the fact that $G^B_{\alpha,i}(z)E\Big[\psi^B_{\alpha,i}(X_i)\Big|\psi^B_{\alpha,i}(X_i) \le z\Big] + \Big(1 - G^B_{\alpha,i}(z)\Big)E\Big[\psi^B_{\alpha,i}(X_i)\Big|\psi^B_{\alpha,i}(X_i) \ge z\Big] = E\Big[\psi^B_{\alpha,i}(X_i)\Big] = (1-\alpha)E[X_i] - E[\eta(X_i)]$, we obtain

$$\alpha\omega_{\alpha,i}(z) = \Big(G^B_{\alpha,i}(z) - G^S_{\alpha,i}(z)\Big)z + \Big(1 - G^B_{\alpha,i}(z)\Big)E\Big[\psi^B_{\alpha,i}(X_i)\Big|\psi^B_{\alpha,i}(X_i) \ge z\Big]$$
$$+ G^S_{\alpha,i}(z)E\Big[\psi^S_{\alpha,i}(X_i)\Big|\psi^S_{\alpha,i}(X_i) \le z\Big] - (1-\alpha)E[X_i] + E[\eta(X_i)].$$

Evaluating at $\underline{z}$ and $\overline{z}$, we find

$$\alpha\omega_{\alpha,i}(\underline{z}) = E\Big[\max\{\underline{z}, \psi^B_{\alpha,i}(X_i)\}\Big] - (1-\alpha)E[X_i] + E[\eta(X_i)]$$
$$= E\Big[\max\{\underline{z} - X_i + \eta(X_i), \psi^B_{\alpha,i}(X_i) - X_i + \eta(X_i)\}\Big] + \alpha E[X_i]$$

$$\text{and } \alpha\omega_{\alpha,i}(\overline{z}) = E\Big[\min\{\psi^S_{\alpha,i}(X_i), \overline{z}\}\Big] - (1-\alpha)E[X_i] + E[\eta(X_i)]$$
$$= E\Big[\min\{\psi^S_{\alpha,i}(X_i) - X_i + \eta(X_i), \overline{z} - X_i + \eta(X_i)\}\Big] + \alpha E[X_i].$$

Consequently,

$$\omega_{\alpha,i}(\underline{z}) = E[X_i] - E\left[\min\left\{\frac{X_i - \eta(X_i) - \underline{z}}{\alpha}, \frac{1 - F_i(X_i)}{f_i(X_i)}\right\}\right] \tag{23}$$

$$\text{and } \omega_{\alpha,i}(\overline{z}) = E[X_i] + E\left[\min\left\{\frac{F_i(X_i)}{f_i(X_i)}, \frac{\overline{z} - X_i + \eta(X_i)}{\alpha}\right\}\right]. \tag{24}$$

Recall that $\underline{z} < x - \eta(x) < \overline{z}$ for all $x \in (0,1)$. Hence, the second expectation on the RHS of (23) and (24) is positive and strictly decreasing in α, yielding $\omega_{\alpha,i}(\underline{z}) < E[X_i] < \omega_{\alpha,i}(\overline{z})$ and implying that $\omega_{\alpha,i}(\underline{z})$ is strictly increasing and $\omega_{\alpha,i}(\overline{z})$ is strictly decreasing in α. Moreover, from (23) and (24), $\lim_{\alpha\to 0}\omega_{\alpha,i}(\underline{z}) = E[X_i] - E\Big[\frac{1-F(X_i)}{f(X_i)}\Big] = 0$ and $\lim_{\alpha\to 0}\omega_{\alpha,i}(\overline{z}) = E[X_i] - E\Big[\frac{F(X_i)}{f(X_i)}\Big] = 1$. □

We are now ready to prove Proposition 3. First note that $r_i = 1$ implies $z_i = \overline{z}$.

Hence, according to Theorem 2, $r_i^* = 1$ is optimal if and only if

$$\omega_{\alpha,i}(\overline{z}) - E[\eta(X_i)] \le \omega_{\alpha,j}(\underline{z}) - E[\eta(X_j)] \quad \text{for all } j \ne i. \tag{25}$$

Part (i): As $\omega_{\alpha,i}(\overline{z})$ is strictly decreasing and $\omega_{\alpha,j}(\underline{z})$ is strictly increasing in α (see Lemma 9), if (25) holds (does not hold) for some $\hat{\alpha}$, then it also holds (does not hold) for all $\alpha > (<)\, \hat{\alpha}$.

Part (ii): Because of Lemma 9, (25) implies $E[X_i - \eta(X_i)] < E[X_j - \eta(X_j)]$, which is equivalent to $E[v_i(\mathbf{X})] < E[v_j(\mathbf{X})]$. Moreover, $1 - E[\eta(X_i)] \le -E[\eta(X_j)]$ is sufficient for (25) to hold since $0 \le \omega_{\alpha,i}(z) \le 1$ for all z.

Part (iii): Suppose $\eta'(x) \in (-1, 1)$ for all $x \in [0, 1]$. Condition (25) can be written as

$$\omega_{\alpha,i}(\overline{z}) - \omega_{\alpha,j}(\underline{z}) \le E[\eta(X_i) - \eta(X_j)] = \int_0^1 \eta'(x)\Big(F_j(x) - F_i(x)\Big)dx < 1.$$

Because $\lim_{\alpha\to 0} \omega_{\alpha,i}(\overline{z}) = 1$ and $\lim_{\alpha\to 0} \omega_{\alpha,j}(\underline{z}) = 0$, (25) is violated for α small enough.

□

B.7 Proof of Proposition 4

Define the function $L(r_1) := \omega_{\alpha,1}(z_1^*) - \omega_{\alpha,2}(z_2^*) - E[\eta(X_1) - \eta(X_2)]$ where $(z_1^*, z_2^*) = \Gamma_2^{-1}(r_1, 1 - r_1)$. From (15), we have $\alpha L(r_1) = -\frac{\partial V_\alpha(r_1)}{\partial r_1}$, so that the concavity of V_α implies that $L(r_1)$ is nondecreasing. According to Theorem 2, an optimal ownership structure $r_1^* = 1 - r_2^*$ satisfies either $r_1^* \in (0, 1)$ and $L(r_1^*) = 0$, or $r_1^* = 0$ and $L(0) \ge 0$, or $r_1^* = 1$ and $L(1) \le 0$.

Recall that $\omega_{\alpha,i}$ is strictly increasing for $i = 1, 2$. Moreover, observe that the characterization of the optimal bilateral dissolution mechanism in Corollary 1 implies that z_1^* is strictly increasing and z_2^* is strictly decreasing in r_1 for all $r_1 \in [0, \underline{r}_1)$ and $r_1 \in (\overline{r}_1, 1]$. Consequently $L(r_1)$ is strictly increasing on $[0, \underline{r}_1)$ and $(\overline{r}_1, 1]$. If the optimal ownership structure r_1^* is such that $L'(r_1^*) > 0$, then r_1^* is unique and so is the corresponding $\mathbf{z}^*$.

We will now show that $L'(r_1) = 0$ for $r_1 \in [\underline{r}_1, \overline{r}_1]$ if and only if also $z_1^* = z_2^* = z$ does not change with r_1 in that range, implying uniqueness of $\mathbf{z}^*$. From Corollary 1 follows that z solves

$$a\Big(G^B_{\alpha,2}(z) + G^S_{\alpha,1}(z)\Big) + (1 - a)\Big(G^S_{\alpha,2}(z) + G^B_{\alpha,1}(z)\Big) = 1$$

for some a that is continuous and strictly increasing in r_1, with $a = 0$ for $r_1 = \underline{r}_1$ and $a = 1$ for $r_1 = \overline{r}_1$. Let the solution to the above equation for a given a be denoted by z^a. If $z^0 = z^1$, then z^a is the same for all a and therefore z does not change with r_1 and

$L'(r_1) = 0$ for $r_1 \in [\underline{r}_1, \overline{r}_1]$. If $z^0 < z^1$ ($z^0 > z^1$), then z^a is strictly increasing (decreasing) in r_1 and

$$G^B_{\alpha,2}(z^a) + G^S_{\alpha,1}(z^a) < (>)\ 1 \quad \text{and} \quad G^S_{\alpha,2}(z^a) + G^B_{\alpha,1}(z^a) > (<)\ 1 \quad \text{for all } a \in (0,1).$$

Using (12), it follows that

$$L'(r_1) = \Big(\omega'_{\alpha,1}(z) - \omega'_{\alpha,2}(z)\Big)\frac{dz}{dr_1} = \frac{1}{\alpha}\Big(G^B_{\alpha,1}(z) - G^S_{\alpha,1}(z) - G^B_{\alpha,2}(z) + G^S_{\alpha,2}(z)\Big)\frac{dz}{dr_1} > 0. \qquad \square$$

B.8 Proof of Lemma 2

First note that because $K \in (\underline{K}, \overline{K}]$, $\alpha^*(\mathbf{r}_K, K) \in (0,1]$ and $W_1(\alpha^*(\mathbf{r}_K, K), \mathbf{r}_K) = K$ for all $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}), \mathbf{r})$. We prove the lemma in two steps. In step 1, we show that each such $\mathbf{r}_K$ is an optimal ownership structure given $\alpha = \alpha^*(\mathbf{r}_K, K)$. In step 2, we show that for all such $\mathbf{r}_K$, $\alpha^*(\mathbf{r}_K, K) = \alpha(K)$, as defined in the lemma.

Step 1: In the following, we will show that if $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}), \mathbf{r})$, then $\mathbf{r}_K \in R^*(\alpha^*(K, \mathbf{r}_K))$. Suppose, by contradiction, $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K,\mathbf{r}), \mathbf{r})$ but $\mathbf{r}_K \notin R^*(\alpha^*_K)$, where $\alpha_K := \alpha^*(K, \mathbf{r}_K) > 0$. Then, there is an $\mathbf{r}' \in R^*(\alpha_K)$ with $\alpha' := \alpha^*(K, \mathbf{r}')$ such that

$$W_0(\alpha_K, \mathbf{r}_K) \geq W_0(\alpha', \mathbf{r}'), \tag{26}$$

$$(1-\alpha_K)W_0(\alpha_K, \mathbf{r}_K) + \alpha_K W_1(\alpha_K, \mathbf{r}_K) < (1-\alpha_K)W_0(\alpha_K, \mathbf{r}') + \alpha_K W_1(\alpha_K, \mathbf{r}') \tag{27}$$

Note that (26) and $\alpha_K > 0$ imply $\alpha' > 0$, leading to $W_1(\alpha_K, \mathbf{r}_K) = W_1(\alpha', \mathbf{r}') = K$. Then, $(1-\alpha')W_0(\alpha', \mathbf{r}_K) + \alpha' W_1(\alpha', \mathbf{r}_K) \geq (1-\alpha')W_0(\alpha_K, \mathbf{r}_K) + \alpha' W_1(\alpha_K, \mathbf{r}_K)$ and (26) yield

$$(1-\alpha')W_0(\alpha', \mathbf{r}_K) + \alpha' W_1(\alpha', \mathbf{r}_K) \geq (1-\alpha')W_0(\alpha', \mathbf{r}') + \alpha' W_1(\alpha', \mathbf{r}'). \tag{28}$$

Because of (27) and (28), we must have $\alpha' \neq \alpha_K$. As the following arguments are essentially the same also for $\alpha' > \alpha_K$, we will restrict attention to $\alpha' < \alpha_K$. Since W_0 and W_1 are continuous in α, (27) and (28) imply that there is an $\alpha'' \in (\alpha', \alpha_K)$ such that

$$(1-\alpha'')W_0(\alpha'', \mathbf{r}_K) + \alpha'' W_1(\alpha'', \mathbf{r}_K) = (1-\alpha'')W_0(\alpha'', \mathbf{r}') + \alpha'' W_1(\alpha'', \mathbf{r}').$$

Note that as W_1 is strictly increasing in α, $W_1(\alpha'', \mathbf{r}_K) < K < W_1(\alpha'', \mathbf{r}')$. Because $W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$ is concave in $\mathbf{r}$ (cf. Theorem 2), there is an $\mathbf{r}_\lambda = \lambda \mathbf{r}^*_K + (1-\lambda)\mathbf{r}'$ for some

$\lambda \in (0,1)$ such that $W_1(\alpha'', \mathbf{r}_\lambda) = K$ and

$$(1-\alpha'')W_0(\alpha'', \mathbf{r}_\lambda) + \alpha'' W_1(\alpha'', \mathbf{r}_\lambda) \geq (1-\alpha'')W_0(\alpha'', \mathbf{r}_K) + \alpha'' W_1(\alpha'', \mathbf{r}_K).$$

Moreover, since $W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha) > W_\alpha(\mathbf{s}^{\mathbf{r}}_{\tilde{\alpha}}, \mathbf{t}^{\mathbf{r}}_{\tilde{\alpha}})$ for all $\tilde{\alpha} \neq \alpha$,

$$(1-\alpha'')W_0(\alpha'', \mathbf{r}_K) + \alpha'' W_1(\alpha'', \mathbf{r}_K) > (1-\alpha'')W_0(\alpha_K, \mathbf{r}_K) + \alpha'' W_1(\alpha_K, \mathbf{r}_K).$$

Recalling $W_1(\alpha_K, \mathbf{r}_K) = K = W_1(\alpha'', \mathbf{r}_\lambda)$, we hence obtain $W_0(\alpha'', \mathbf{r}_\lambda) > W_0(\alpha_K, \mathbf{r}_K)$, which contradicts $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K, \mathbf{r}), \mathbf{r})$.

Step 2: Consider α and $\mathbf{r} \in R^*(\alpha)$ such that $W_1(\alpha, \mathbf{r}) = K$. Then, for any $\alpha' \neq \alpha$ and $\mathbf{r}' \in R^*(\alpha')$,

$$\begin{aligned}(1-\alpha)W_0(\alpha, \mathbf{r}) + \alpha W_1(\alpha, \mathbf{r}) &\geq (1-\alpha)W_0(\alpha, \mathbf{r}') + \alpha W_1(\alpha, \mathbf{r}') \\ &> (1-\alpha)W_0(\alpha', \mathbf{r}') + \alpha W_1(\alpha', \mathbf{r}'),\end{aligned}$$

where the second line follows from $W_\alpha(\mathbf{s}^{\mathbf{r}'}_\alpha, \mathbf{t}^{\mathbf{r}'}_\alpha) > W_\alpha(\mathbf{s}^{\mathbf{r}'}_{\tilde{\alpha}}, \mathbf{t}^{\mathbf{r}'}_{\tilde{\alpha}})$ for all $\tilde{\alpha} \neq \alpha$. Similarly,

$$\begin{aligned}(1-\alpha')W_0(\alpha, \mathbf{r}) + \alpha' W_1(\alpha, \mathbf{r}) &< (1-\alpha')W_0(\alpha', \mathbf{r}) + \alpha W_1(\alpha', \mathbf{r}) \\ &\leq (1-\alpha')W_0(\alpha', \mathbf{r}') + \alpha' W_1(\alpha', \mathbf{r}').\end{aligned}$$

Combining the inequalities and rearranging yields

$$\frac{1-\alpha'}{\alpha'}\Big(W_0(\alpha, \mathbf{r}) - W_0(\alpha', \mathbf{r}')\Big) < W_1(\alpha', \mathbf{r}') - K < \frac{1-\alpha}{\alpha}\Big(W_0(\alpha, \mathbf{r}) - W_0(\alpha', \mathbf{r}')\Big).$$

Hence, $W_1(\alpha', \mathbf{r}') \neq K$. Consequently, there exists a unique α such that $W_1(\alpha, \mathbf{r}) = K$ for some $\mathbf{r} \in R^*(\alpha)$, which is defined as $\alpha(K)$ in the lemma. Step 1 now implies that if $\mathbf{r}_K \in \arg\max_{\mathbf{r}\in\Delta^{n-1}} W_0(\alpha^*(K, \mathbf{r}), \mathbf{r})$, then $\mathbf{r}_K \in R^*(\alpha(K))$. □

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